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ON FROBENIUSEAN ALGEBRAS. II

BY TADASI NAKAYAMA

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Introduction

In the present paper we continue our study of a class of (associative) algebras called Frobeniusean.[1] Chapter I, which is short, is a direct sequel of Part I and deals chiefly with a certain type of automorphisms in a Frobeniusean algebra. The automorphisms enable us to generalize and refine some of the theorems in Part I, and also clarify the significance of symmetric algebras. Further, an application of our results to the theory of Galois moduli and normal bases over modular fields is discussed. In Chapter II we make an attempt to extend our theory from algebras to general rings satisfying chain conditions. To do so we adopt the properties of I, 2, Lemma 2 as a definition of Frobeniusean and quasi-Frobeniusean rings. Then many of the theorems in I can be extended to this general case. But since this definition does not have much significance for itself, contrary to the case of algebras, Chapter II may be considered as a study of the structure of those rings in which the annihilation gives a 1-1 correspondence between left and right ideals.[2] Sections 6, 7 aim, however, not only at Frobeniusean rings but at Frobeniusean algebras.

CHAPTER I: A FURTHER STUDY OF FROBENIUSEAN ALGEBRAS

1. A class of automorphisms in a Frobeniusean algebra

Let A be a Frobeniusean algebra over a field F. Let $a_1, a_2, \cdots, a_n$ be a basis of A with a multiplication table

$$a_\rho a_\sigma = \sum_\tau \alpha_{\rho\sigma\tau} a_\tau . \tag{1}$$

There exists a vector (λ_τ) such that the corresponding parastrophic matrix $P = (\sum_\tau \alpha_{\rho\sigma\tau}\lambda_\tau)$ is non-singular. We consider again the linear form $\lambda(x) = \sum \lambda_\rho \xi_\rho$ $(x = \sum \xi_\rho a_\rho \in A)$ and the *non-singular* hyperplane H defined by $\lambda(x) = 0$. For a set S of elements in A we denote by $p_H(S)[q_H(S)]$, as before, the set of all x satisfying $xS \subseteq H[Sx \subseteq H]$. Any other non-singular hyperplane H' in A can be given as $H' = p_H(c^{-1}) = Hc$ with a regular element c, and conversely with any regular element c Hc is a non-singular hyperplane in A (Cf. I, 6).

[1] *On Frobeniusean algebras. I*, Ann. Math. 40 (1939) referred to as Part I, or simply as I.
[2] M. Hall (7) calls them (weakly) closed rings.

Theorem 1. *There exists a uniquely determined automorphism* φ: $x \to x^*$ *of* A *such that*

$$\lambda(x^*y) = \lambda(yx) \qquad (\text{that is, } x^*y - yx \in H) \tag{2}$$

for all $y \in A$. H *remains invariant under* φ; $H^* = H$.

φ*'s corresponding to different choices of non-singular hyperplanes in* A *form a co-set with respect to the invariant subgroup consisting of all inner automorphisms in the automorphism group of* A.

Proof. If $x = \sum \xi_\rho a_\rho$ we put $(\xi_\rho^*) = P(P')^{-1}(\xi_\rho)$ and $x^* = \sum \xi_\rho^* a_\rho$. Then[3] $\lambda(x^*y) = (\xi^*)P(\eta) = (\xi)P'(\eta) = (\eta)P(\xi) = \lambda(yx)$ where $y = \sum \eta_\rho a_\rho$. This shows the existence of a mapping φ: $x \to x^*$ such that (2) is true for all $x, y \in A$. φ is uniquely determined by this property as we can see easily by reversing the above computation and observing particularly that P is non-singular. The mapping is evidently 1-1, and that $(\alpha x + \beta z)^* = \alpha x^* + \beta z^*$ is obvious. Moreover, the relation $\lambda(x^*z^*y) = \lambda(z^*yx) = \lambda(yxz)$ shows that $(xz)^* = x^*z^*$. Hence φ is an automorphism. Now, $x \in H$ implies $x^* \in H$, since $x^* - x = x^*1 - 1x \in H$. Thus $H^* = H$.

To prove the second part of the theorem, consider a second non-singular hyperplane $H' = Hc$ (c regular). Since $(cxc^{-1})^*y - yx = ((cxc^{-1})^*yc^{-1} - yc^{-1}(cxc^{-1}))c \in Hc = H'$ for all $x, y \in A$, the automorphism φ' corresponding to H' is given by $x \to (cxc^{-1})^*$.

Lemma 1. *The automorphism* φ *maps* $q_H(S)$, *with any set* S *in* A, *onto* $p_H(S)$; $q_H(S)^* = p_H(S)$. *In particular* $r(\mathfrak{l})^*(= r(\mathfrak{l}^*)) = p_H(\mathfrak{l})$ *for a left ideal* $\mathfrak{l}$ *in* A.[4]

Proof. Immediate from the definition and I, Lemma 4.

Thus the automorphism φ *correlates the two kinds of dual correspondences between left and right ideals in* A; one is given by annihilation and the other is representation-theoretical. Namely

Theorem 2.[5] *If* $\mathfrak{l}$, $\mathfrak{l}_0$ *are left ideals in* A *and* $\mathfrak{l} \supseteqq \mathfrak{l}_0$, *the representation of* A *defined by the left module* $\mathfrak{l}/\mathfrak{l}_0$ *is equivalent to the one defined by the right module* $r(\mathfrak{l}_0)^*/r(\mathfrak{l})^*$. *Furthermore, the representation defined by a left principal ideal* Ac *is equivalent to the one defined by the right ideal* c^*A.

Proof. The theorem follows from the above lemma and I, 8, Lemma 6. For the second half cf. also the proof of I, 5, Th. 4.

Theorem 3. *The automorphism algebra of the left module* $\mathfrak{l}/\mathfrak{l}_0[Ac]$ *is inversely isomorphic to that of the right module* $r(\mathfrak{l}_0)/r(\mathfrak{l})[cA]$.[6]

Proof. The automorphism algebra of a left [right] module is inversely [directly] isomorphic to the algebra of matrices which commute with all the matrices in a representation belonging to the module. It follows thus from Theorem 2 that the automorphism algebra of $\mathfrak{l}/\mathfrak{l}_0$ is inversely isomorphic to

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[3] Consult the form of the parastrophic matrix P. Cf. also I, 3.

[4] $r(S)$ denotes, as before, the set of right annihilators of S in A.

[5] Cf. Th. 13 in the section 6 below.

[6] The last statement of I, Th. 5 is a very special case of the present theorem.

that of $r(\mathfrak{l}_0)^*/r(\mathfrak{l})^*$. But this latter is evidently isomorphic to the automorphism algebra of $r(\mathfrak{l}_0)/r(\mathfrak{l})$.

Now, let E_κ, $e_{\kappa,i}$ ($\kappa = 1, 2, \cdots, k$; $i = 1, 2, \cdots, f(\kappa)$) and $\pi(\kappa)$ ($\kappa = 1, 2, \cdots, k$) have the same significance as in I. In particular $U_\kappa \cong V_{\pi(\kappa)}$. Then

THEOREM 4. $E^*_{\pi(\kappa)} = E_\kappa$. *With a suitable choice of H we have furthermore* $e^*_{\pi(\kappa),i} = e_{\kappa,i}$ *(for all κ, i).*[7]

PROOF. Obviously $E^*_{\pi(\kappa)} = E_\lambda$ for a certain λ. But the representation defined by $E_\lambda A$ must be equivalent to the one defined by $AE_{\pi(\kappa)}$. Hence $\lambda = \kappa$. To prove the second half we observe that we can choose H so that $e_{\kappa,i}Ae_{\lambda,j} \subseteq H$ if $(\pi(\kappa), i) \neq (\lambda, j)$; this can be seen in exactly the same way as in Nakayama-Nesbitt (11). With such a choice of H we have indeed $e^*_{\pi(\kappa),i} = e_{\kappa,i}$. For, $\lambda(e_{\kappa,i}x) = \lambda(\sum_{\lambda,j} e_{\kappa,i}xe_{\lambda,j}) = \lambda(e_{\kappa,i}xe_{\pi(\kappa),i}) = \lambda(\sum_{\lambda,j} e_{\lambda,j}xe_{\pi(\kappa),i}) = \lambda(xe_{\pi(\kappa),i})$.

Another immediate consequence of Lemma 1 is that if $\mathfrak{z}$ is a two-sided ideal in A then

$$l(\mathfrak{z}) = r(\mathfrak{z})^* = r(\mathfrak{z}^*).$$

In particular if $\mathfrak{z} = \mathfrak{z}^*$, or if more particularly $\mathfrak{z}$ is invariant under all automorphisms of A, then $l(\mathfrak{z}) = r(\mathfrak{z})$. (This proves again I, Th. 6 in case where the algebra is not only quasi-Frobeniusean but Frobeniusean.) We find moreover that we can put $d = c^*$ in I, Th. 9, Corollary.

We note further that if $S(x)$ and $R(x)$ denote, as before, the left and the right regular representations of A with respect to the basis $a_1, a_2, \cdots, a_n$ then we have the relation

$$R(x)P' = P'S(x^*)$$

as a counterpart of the fundamental relation $R(x)P = PS(x)$. For, $(a_1^*, \cdots, a_n^*) = (a_1, \cdots, a_n)(P')^{-1}P$, as can be seen from the proof of Theorem 1, and if we put $\bar{R}(x) = R(\varphi^{-1}(x))$ then $x \rightarrow \bar{R}(x)$ is the right regular representation with respect to the basis (a_i^*), whence $\bar{R}(x) = P'P^{-1}R(x)(P'P^{-1})^{-1} = P'S(x)(P')^{-1}$. But this is nothing but the above relation.

For an interesting application of the automorphism φ to a generalization of the orthogonality relation among representation coefficients, see a forthcoming paper by C. Nesbitt.

REMARK. Our algebra A is symmetric if and only if the automorphism φ is inner. The above treatment of Frobeniusean algebras can be considered as an extension of the section 9 in I.

EXAMPLE. *Cartan's algebra of outer multiplication*[8] is Frobeniusean. Namely, let m be any natural number and consider an algebra A over a field F which is generated by F and m elements $c_1, c_2, \cdots, c_m$ and in which the law of compo-

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[7] The automorphism φ^{-1} effects the permutation π, so to speak.

[8] See for instance Cartan (19). This example was suggested to the writer by C. Chevalley.

sition is given by $c_i^2 = 0$, $c_ic_j = -c_jc_i$ for $i \neq j$. This algebra A, of Cartan's outer multiplication, has a basis 1, $c_{i_1}c_{i_2} \cdots c_{i_s}$ ($i_1 < i_2 < \cdots < i_s$), and is Frobeniusean. For, if H is the hyperplane in A consisting of those elements which have a vanishing coefficient for the term $c_1c_2 \cdots c_m$ when expressed by the above basis, then evidently H does not contain any ideal except the zero ideal. The automorphism φ belonging to the same H is defined simply by $c_i \to c_i^* = (-1)^{m-1}c_i$. If the characteristic of F is different from 2, then A is symmetric or not according as m is odd or even.[9]

2. Galois moduli over modular fields

Let K be a (finite and separable) Galois extension over F, and let $\mathfrak{G}$ be the Galois group of K/F. Looking upon the elements of $\mathfrak{G}$ as the right operators on K, we can consider K/F as a *right* representation module of the group algebra $\mathfrak{G}(F)$ of $\mathfrak{G}$ over F. The well-known theorem of normal bases[10] states that this module is (operator-) isomorphic with $\mathfrak{G}(F)$ itself, that is, K/F defines a representation of $\mathfrak{G}$ equivalent to the regular representation. Furthermore, the image of a *left* ideal of $\mathfrak{G}(F)$ by such an (operator-) isomorphism between $\mathfrak{G}(F)$ and K/F is independent of the special choice of isomorphism, and such an image of a left ideal of $\mathfrak{G}(F)$ in K is called a *Galois module* of K/F.[11]

Let K_1 be a field between F and K; $K \supseteqq K_1 \supseteqq F$, and let $\mathfrak{H}$ be the subgroup of $\mathfrak{G}$ belonging to K_1. The group algebra $\mathfrak{H}(K_1)$ of $\mathfrak{H}$ over K_1 is then (operator-) isomorphic to K/K_1 with respect to the right operator algebra $\mathfrak{H}(K_1)$, and Galois moduli of K/K_1 are defined in the same way as above. Now, the following interesting theorem was proved in Deuring (21) under the assumption that the group algebra $\mathfrak{G}(F)$ is semisimple or, what is equivalent, that the degree $(K:F)$ is not divisible by the characteristic of F:

THEOREM 5. *Let $\mathfrak{m}$ be a Galois module of K/F such that it is a K_1-module. Then $\mathfrak{m}$ is also a Galois module of K/K_1, and moreover, the representation of $\mathfrak{G}$ obtained from the left ideal of $\mathfrak{G}(F)$ corresponding to the K/F-Galois-module $\mathfrak{m}$ is equivalent to the one induced from the representation of the subgroup $\mathfrak{H}$ obtained from the left ideal of $\mathfrak{H}(K_1)$ corresponding to the K/K_1-Galois-module $\mathfrak{m}$.*

The purpose of the present section is to note that this theorem holds *without assuming the semisimplicity of $\mathfrak{G}(F)$.*

PROOF. Let $\mathfrak{l}$ be the left ideal of $\mathfrak{G}(F)$ which corresponds to our K/F-Galois-

[9] A can be considered also as the algebra of chains contained in an (absolute) simplex spanned by m vertices $c_1, c_2, \cdots, c_m$.

[10] Noether (26), Deuring (20), (21), Brauer (18). Artin has given another simple and elegant proof.

[11] Deuring (21). It seems to the writer that there is a slight confusion in the usage of the term "Galois module." Noether defined first a Galois module as an image of a right ideal. Deuring used *essentially* the same definition in his paper (20), but switched in his second paper (21), as well as in his book (5), to the definition which we are adopting in the present paper.

module $\mathfrak{m}$. $\mathfrak{l}$ consists of all left annihilators of $\mathfrak{r} = r(\mathfrak{l})$ (I, Th. 1. Cf. also I, 9), and thus Deuring's method can be transferred term by term to our general case. Namely, $\mathfrak{m}$ consists of all elements a in K satisfying

$$\sum_S a^S \beta_S = 0 \quad \text{for all} \quad \sum_S S\beta_S \in \mathfrak{r}. \tag{3}$$

Due to our assumption that $\mathfrak{m}$ is a K_1-module, we have

$$\sum_S h^S a^S \beta_S = \sum_T h^T \sum_{P \in \mathfrak{H}} a^{PT} \beta_{PT} = 0$$

for any h in K_1, where $\mathfrak{G} = \sum_T \mathfrak{H} T$ is the right co-set decomposition of $\mathfrak{G}$ with respect to $\mathfrak{H}$. Applying this relation to a system of basis elements h_1, $h_2, \cdots, h_m$ of K_1/F and noticing that the discriminant $| h_i^T |^2 \neq 0$, we find $\sum_P a^{PT} \beta_{PT} = 0$, or

$$\sum_P a^P \beta_{PT} = 0 \qquad \left(\sum_S S\beta_S \in \mathfrak{r}\right) \tag{4}$$

for all T (mod $\mathfrak{H}$). Since conversely (4) implies (3), the elements a of $\mathfrak{m}$ can be characterized also by (4). Hence the image of $\mathfrak{m}$ by an isomorphism between K/K_1 and $\mathfrak{H}(K_1)$ consists of all left annihilators of the elements $\sum_P P\beta_{PT}$ ($\sum_S S\beta_S \in \mathfrak{r}$), and is a left ideal in $\mathfrak{H}(K_1)$. Hence $\mathfrak{m}$ is a Galois module of K/K_1. The second half of the theorem can be proved also in the same way as in the original paper of Deuring (p. 46).[12]

3. Appendix: On normal bases

Deuring's second proof of the theorem of normal bases, which was published in the same paper Deuring (21) also under the assumption that the characteristic does not divide the degree of extension, can be so modified that it works generally. Furthermore, R. Stauffer's[13] method for constructing a normal basis works, after a modification, also without that same assumption. These will be seen in the following.

Let $\mathfrak{G}$ be, as above, the Galois group of a Galois extension K/F. On taking a sufficiently large over-field F' of F such that all absolutely irreducible representations of $\mathfrak{G}$ lie in F', we consider the algebra $K' = K \times F'$ over F' instead of K itself. It is a right representation module of $\mathfrak{G}$ and we denote by $M(S)$ ($S \in \mathfrak{G}$) the corresponding representation of $\mathfrak{G}$. Let $G(S)$ be an irreducible representation of $\mathfrak{G}$ (in F') with a degree, say g, and let $U(S)$ be the corresponding directly indecomposable part of the regular representation of $\mathfrak{G}$; $U(S)$ has $G(S)$ as its first and its last irreducible as well as largest completely reducible components. We wish to show that $U(S)$ is contained in $M(S)$ (at least) g times.—If this is proved for each irreducible representation $G(S)$, then we find that the regular representation of $\mathfrak{G}$ is contained in, whence coincides

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[12] Theorem remains true if we, as in Deuring (21), extend the underlying field F to an over-field F' and consider $K \times F'$, $K_1 \times F'$ instead of K, K_1.

[13] Stauffer (28).

with, $M(S)$ and thus we are through.[14]—That $U(S)$ is contained in $M(S)$ g times is equivalent to that there exists in K' a matrix B with g columns, satisfying

$$B^S = U(S)B \quad \text{for all} \quad S \in \mathfrak{G} \tag{5}$$

and whose elements are all linearly independent (with respect to F'). Now, if we denote by B_1 the matrix consisting of the first g rows of a matrix B satisfying (5), then B_1 fulfills $B_1^S = G(S)B_1$ for all S.[15] Furthermore, since the representation module of $U(S)$ possesses a *unique* simple submodule (corresponding to $G(S)$), it follows easily that in order to show that all the elements of B are linearly independent we have merely to show that the elements of B_1 are linearly independent. And, in order to secure the latter we have in turn only to make sure that all the columns of B_1 are linearly independent (with respect to F'); this is due to the fundamental fact that every automorphism of an (irreducible) representation module of $G(S)$ can be obtained by multiplication with an element of F' (cf. Deuring (21), pp. 43–44 and Stauffer (28), pp. 592–593). Hence our aim is to prove the existence of B with g columns, satisfying (5) and such that the g columns of B_1 are linearly independent with respect to F'.

For this purpose, we can first apply *Speiser's theorem.* (In a *generalized* form the theorem asserts: Let $M_E, M_S, \cdots, M_T$ and $N_E, N_S, \cdots, N_T$ ($\mathfrak{G} = \{E, S, \cdots, T\}$) be two systems of non-singular matrices with elements from K' such that

$$M_S^T M_T M_{ST}^{-1} = N_S^T N_T N_{ST}^{-1}$$

for every pair $S, T \in \mathfrak{G}$. Then there exists a non-singular matrix C which satisfies $C^S M_S C^{-1} = N_S$ for all $S \in \mathfrak{G}$.)[16] Namely, there is a non-singular square matrix C fulfilling $C^S C^{-1} = U(S)$ (that is, $C^S = U(S)C$) for all S. We choose g columns in such a C so that the square matrix C_0 consisting of g^2 elements at the intersections of those g columns with the first g rows is non-singular; this is obviously possible. Denote by B the matrix consisting of the chosen g columns of C. Then this B satisfies the above requirements, because (5) is evidently valid and the corresponding B_1 is simply C_0 whose g columns are linearly independent even with respect to K'.

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[14] If a representation contains $U(S)$ then $U(S)$ is a direct constituent. See Nakayama-Nesbitt (11), §2.

[15] Here we agree that $U(S)$ has already assumed the form where the right upper part is 0.

[16] This can be proved in exactly the same manner as in the special case Schur (27) (or Weyl (30)), provided that F' contains sufficiently many elements. The case where this last condition is not satisfied can be reduced to a favorable case by an argument due to R. Brauer and E. Noether (cf. Deuring (20) or Van der Waerden (29), p. 70).

The present form, which is more general than needed here and which is also more general than the one given in Deuring (21), for instance, has a significance for the classification of semi-linear transformations; cf. Nakayama (25), Haantjes (22).

A second method to obtain a relevant B is to consider a matrix

$$B = B(z) = \sum_S \tilde{U}(S^{-1})z^S, \qquad z \in K,$$

where by $\tilde{U}(S^{-1})$ we denote the matrix composed of the first g columns of $U(S^{-1})$. This $B = B(z)$ certainly satisfies (5), and the corresponding B_1 is $B_1(z) = \sum_S G(S^{-1})z^S$. Now, $B_1(z)$ satisfies not only the relation $B_1(z)^S = G(S)B_1(z)$ but also the relation

$$B_1(z^S) = B_1(z)G(S).$$

From this fact we can conclude that all the elements of $B_1(z)$ are linearly independent (with respect to F') whenever one of them is different from 0, that is, whenever $B_1(z) \neq 0$. But from the separability of K/F we can see easily that there exists certainly a $z \in K$ such that $B_1(z) \neq 0$. (For all this cf. Stauffer (28)). Thus $B = B(z)$ with such a z possesses the desired properties.

Furthermore, let $G^{(1)}, G^{(2)}, \cdots, G^{(m)}$ be the totality of distinct irreducible representations of $\mathfrak{G}$ in F', and take for each $G(S) = G^{(\mu)}(S)$ an element $z = z^{(\mu)}(\in K)$ satisfying the above condition. Here we can, and shall, choose these $z^{(\mu)}$ in such a manner that $z^{(\mu)} = z^{(\nu)}$ if $G^{(\mu)}$ and $G^{(\nu)}$ are conjugate with respect to F. Let now $G_1, G_2, \cdots, G_k$ be the totality of distinct irreducible representations of $\mathfrak{G}$ in F, and put $z_\kappa = z^{(\mu)}$ if G_κ contains $G^{(\mu)}$. Let further $E_1, E_2, \cdots, E_k$ be mutually orthogonal idempotent elements in the group algebra $\mathfrak{G}(F)$ with the sum $\sum E_\kappa = E = 1$ such that if N denotes the radical of $\mathfrak{G}(F)$ then E_κ (mod N) is the unit element in the simple two-sided ideal of $\mathfrak{G}(F)/N$ belonging to $\mathfrak{G}_\kappa$. ($E_\kappa\mathfrak{G}(F)$ is a direct sum of right ideals which define the component U_κ of the regular representation corresponding to G_κ.) Then $w = \sum_\kappa z_\kappa^{E_\kappa}(\in K)$ and its conjugates form a normal basis of K/F. This can be seen in quite a similar way as in Stauffer (28).

Chapter II: Frobeniusean Rings

4. Frobeniusean and quasi-Frobeniusean rings

Consider a ring A. A may have a left, say, operator domain Ω such that $\alpha(a + b) = \alpha a + \alpha b$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$ if $\alpha \in \Omega$ and $a, b \in A$. In case A satisfies both the minimum and the maximum conditions for left and right ideals (allowable with respect to Ω), its general structure is well known. But recently C. Hopkins showed that the theory remains valid to a large extent in the case where A fulfills the minimum condition only.[17] We assume in the present treatment also merely the *minimum condition*. And, since we shall deal exclusively with such type of rings, we shall understand by a *ring* always a one of the type.

A possesses the radical N which is nilpotent and the residue class ring $\bar{A} = A/N$ is semisimple. Everything in Part I, 1 remains valid[18] for our A

[17] Hopkins (23), (24).

[18] With trivial modifications of terminologies, of course.

except the second half of Lemma 1 and the statements concerning the completely reducible ideals. Thus we retain the old significances of the symbols k, $f(\kappa)$, $e_{\kappa,i}$, $e_\kappa(=e_{\kappa,1})$, $E_\kappa=\sum_{i=1}^{f(\kappa)}e_{\kappa,i}$, $E=\sum_{\kappa=1}^{k}E_\kappa$, $c_{\kappa,ij}$ $(\kappa=1,2,\cdots,k;$ $i,j=1,2,\cdots,f(\kappa))$. Namely

$$\bar{A}=\bar{A}_1+\bar{A}_2+\cdots+\bar{A}_k,$$

where each $\bar{A}_\kappa$ is simple and has a unit element $\bar{E}_\kappa=E_\kappa \pmod N$; $e_{\kappa,i}$ are mutually orthogonal idempotent elements, and $c_{\kappa,ii}=e_{\kappa,i}$, $c_{\kappa,ij}c_{\lambda,hl}=\delta_{\kappa\lambda}\delta_{jh}c_{\kappa,il}$. As to the connection between $r(N)$ and completely reducible left ideals, we can assert that any completely reducible left ideal is contained in $r(N)$, while $r(N)$ is a direct sum of a completely reducible left ideal and a left ideal which is annihilated by A; this last left ideal is not necessarily completely reducible this time.[19]

Remark. If A possesses a *unit element* then it satisfies also the maximum condition for left and right ideals.[20] Moreover, the same assumption assures of course that $r(N)$ is really the largest completely reducible left ideal.

If $\mathfrak{m}$ is a simple left module of A, then we denote by $d_l(\mathfrak{m})$ the rank of $\mathfrak{m}$ with respect to the quasi-field of automorphisms. Hence, if $\mathfrak{m}\cong\bar{A}\bar{e}_\kappa=Ae_\kappa/Ne_\kappa$ then $d_l(\mathfrak{m})=f(\kappa)$. More generally, if $\mathfrak{m}$ is a left module which has a composition series $\mathfrak{m}=\mathfrak{m}_0\supset\mathfrak{m}_1\supset\cdots\supset\mathfrak{m}_s\supset 0$, then we put $d_l(\mathfrak{m})=\sum_{i=1}^{s}d_l(\mathfrak{m}_{i-1}/\mathfrak{m}_i)$. For a right module $\mathfrak{n}$ possessing a composition series we define $d_r(\mathfrak{n})$ in the same manner.

Now, since a direct generalization of the definition of Frobeniusean and quasi-Frobeniusean algebras to our general case seems difficult, let us, in view of I, Lemma 2, define the corresponding types of rings as follows:

Definition. A is called *quasi-Frobeniusean* if it possesses a unit element and if there exists a permutation $(\pi(1),\pi(2),\cdots,\pi(k))$ of $(1,2,\cdots,k)$ such that for each κ

i) $e_\kappa A$ has a unique simple right subideal $\mathfrak{r}_\kappa$ and $\mathfrak{r}_\kappa\cong\bar{e}_{\pi(\kappa)}\bar{A}$,

ii) $Ae_{\pi(\kappa)}$ has a unique simple left subideal $\mathfrak{l}_{\pi(\kappa)}$ and $\mathfrak{l}_{\pi(\kappa)}\cong\bar{A}\bar{e}_\kappa$.

If moreover

iii) $f(\kappa)=f(\pi(\kappa))$,

then we call A *Frobeniusean.*

Remark.[20a] In the above definition we can omit the condition $\mathfrak{l}_{\pi(\kappa)}\cong\bar{A}\bar{e}_\kappa$ in ii). For, if the other conditions are satisfied, that is, if A has a unit element and if i) and the first part of ii) are the case, then $e_\kappa\mathfrak{l}_{\pi(\kappa)}=e_\kappa r(N)e_{\pi(\kappa)}\supseteqq\mathfrak{r}_\kappa e_{\pi(\kappa)}\neq 0$, whence necessarily $\mathfrak{l}_{\pi(\kappa)}\cong\bar{A}\bar{e}_\kappa$.

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[19] This is due to our not assuming anything about the structure of Ω.

[20] See Hopkins (24). As a matter of fact, our main interest in the present paper lies in such an A which, either by its definition or as a consequence of its definition, possesses a unit element. Hence our present avoidance of maximum condition is not so important, according to this remark.

[20a] See a correction at the end (Added in proof).

We have now the following theorems which correspond to Th. 1, 2 and 3 in Part I:

THEOREM 6. *If a ring A is quasi-Frobeniusean, then*

$$\alpha) \qquad l(r(\mathfrak{l})) = \mathfrak{l}, \qquad r(l(\mathfrak{r})) = \mathfrak{r}$$

for every left ideal $\mathfrak{l}$ and right ideal $\mathfrak{r}$. Conversely, if α) holds for every nilpotent simple left ideal $\mathfrak{l}$ and nilpotent simple right ideal $\mathfrak{r}$ as well as for $\mathfrak{l} = \mathfrak{r} = N$ (radical of A) and $\mathfrak{l} = \mathfrak{r} = 0$, then A is quasi-Frobeniusean.

THEOREM 7. *If A is Frobeniusean, then besides α) we have*

$$\beta') \qquad d_l(\mathfrak{l}) = d_r(A/r(\mathfrak{l})), \qquad d_r(\mathfrak{r}) = d_l(A/l(\mathfrak{r}))$$

for every left ideal $\mathfrak{l}$ and right ideal $\mathfrak{r}$.[21] *Conversely, if α) is valid for every nilpotent simple left ideal $\mathfrak{l}$, nilpotent simple right ideal $\mathfrak{r}$, $\mathfrak{l} = \mathfrak{r} = N$ and $\mathfrak{l} = \mathfrak{r} = N$ while β') is the case for every nilpotent simple left ideal $\mathfrak{l}$ and nilpotent simple right ideal $\mathfrak{r}$, then A is Frobeniusean.*[22]

PROOF. The second parts of the theorems can be proved in the same way as in I (with a very slight and trivial modification).[23]

To prove the first parts, assume that A is quasi-Frobeniusean. Since $l(N)$ is a two-sided ideal we have $l(N) = \sum_\kappa E_\kappa l(N) = \sum_{\kappa,i} e_{\kappa,i}l(N)$. From the definition of a quasi-Frobeniusean ring it follows that for each κ, i the right ideal $e_{\kappa,i}l(N) = e_{\kappa,i}A \cap l(N)$ is simple and is isomorphic to $\bar{e}_{\pi(\kappa)}\bar{A}$. Hence $e_{\kappa,i}l(N) = e_{\kappa,i}l(N)E_{\pi(\kappa)}$, whereas $e_{\kappa,i}l(N)E_\lambda = 0$ if $\lambda \neq \pi(\kappa)$. Thus $E_\kappa l(N)E_\lambda = E_\kappa l(N)$ or $= 0$ according as $\lambda = \pi(\kappa)$ or not. Since this is the case for every κ, we have $l(N)E_{\pi(\kappa)} = \sum_\mu E_\mu l(N)E_{\pi(\kappa)} = E_\kappa l(N)E_{\pi(\kappa)} = E_\kappa l(N)$. This shows that $E_\kappa l(N)$ is two-sided. Moreover, it is a simple two-sided ideal. To see this, let d be any non-zero element in $E_\kappa l(N)$. $d = \sum_{i=1}^{f(\kappa)} e_{\kappa,i}d$ and at least one of $e_{\kappa,i}d$ is not zero. Suppose $e_{\kappa,p}d \neq 0$. Then $e_{\kappa,p}dA = e_{\kappa,p}l(N)$ since $e_{\kappa,p}l(N)$ is a simple right ideal, and therefore $AdA = Ae_{\kappa,p}l(N) \supseteqq E_\kappa l(N)$. Hence the two-sided ideal $E_\kappa l(N)$ is simple. In particular, it is completely reducible as a left ideal too, that is, $E_\kappa l(N) \subseteqq r(N)$. Because this is true for every κ, it follows that $l(N) \subseteqq r(N)$. But the inclusion of the other direction can be seen in the same way, and we have $l(N) = r(N)$. We denote this two-sided ideal by M.

Let $\mathfrak{l}$ be any non-zero left ideal of A and let $\mathfrak{l}'$ be a maximal left subideal of $\mathfrak{l}$. Suppose $\mathfrak{l}/\mathfrak{l}' \cong \bar{A}\bar{e}_\kappa$. If $c \in A$ then the left ideal $\mathfrak{l}c$ is isomorphic to $\mathfrak{l}/\mathfrak{l} \cap l(c)$, as can be seen from the mapping $b \to bc$ ($b \in \mathfrak{l}$). If in particular $c \in r(\mathfrak{l}')$ then $\mathfrak{l} \cap l(c) \supseteqq \mathfrak{l} \cap l(r(\mathfrak{l}')) \supseteqq \mathfrak{l}'$ whence $\mathfrak{l}c \cong \bar{A}\bar{e}_\kappa$ or $= 0$. This shows that $\mathfrak{l}r(\mathfrak{l}') \subseteqq M$. Let now b_1 be an element in $e_\kappa\mathfrak{l}$ which is not contained in $\mathfrak{l}'$; the existence of such

[21] In particular, we have $d_l(A) = d_r(A)$.

[22] It is justified to speak of d_l and d_r here, since the validity of α) for nilpotent simple ideals as well as for N and 0 implies that A is quasi-Frobeniusean and in particular that A satisfies, because of the existence of a unit element, the maximum condition too.

[23] One has to modify slightly and in an obvious manner the very last part of i) there, while the final step viii) becomes a mere triviality this time.

a b_1 follows from $\mathfrak{l}/\mathfrak{l}' \cong \bar{A}\bar{e}_\kappa$. Then $Ab_1 \cup \mathfrak{l}' = \mathfrak{l}$. From the above consideration we have $b_1 r(\mathfrak{l}') \subseteq e_\kappa \mathfrak{l} r(\mathfrak{l}') \subseteq e_\kappa M$ whence $b_1 r(\mathfrak{l}') = e_\kappa M$ or $= 0$. Thus $b_1 r(\mathfrak{l}') \cong \bar{e}_{\pi(\kappa)}\bar{A}$ or $= 0$. On the other hand $b_1 r(\mathfrak{l}') \cong r(\mathfrak{l}')/r(\mathfrak{l}') \cap r(b_1)$ and here $r(\mathfrak{l}') \cap r(b_1) = r(\mathfrak{l}') \cap r(Ab_1) = r(\mathfrak{l}' \cup Ab_1) = r(\mathfrak{l})$. Hence finally $r(\mathfrak{l}')/r(\mathfrak{l}) \cong \bar{e}_{\pi(\kappa)}\bar{A}$ or $= 0$.

Consider a composition series of left ideals $\mathfrak{l}_0 = 0 \subset \mathfrak{l}_1 \subset \mathfrak{l}_2 \subset \cdots \subset \mathfrak{l}_s = A$ of A. The above observation shows that each right module $r(\mathfrak{l}_t)/r(\mathfrak{l}_{t+1})$ is either simple or 0. Since $r(0) = A$ and $r(A) = 0$, it follows that the length of a composition series of right ideals of A is less than or equal to s, the length of a composition series of left ideals. But the inequality of the other direction can be seen in the same manner. Hence the lengths of composition series of right and left ideals are equal to each other, and moreover the right moduli $r(\mathfrak{l}_t)/r(\mathfrak{l}_{t+1})$ $(t = 0, 1, \cdots, s - 1)$ are all simple. Further, all the left moduli $l(r(\mathfrak{l}_{t+1}))/l(r(\mathfrak{l}_t))$ must be also simple, and necessarily $l(r(\mathfrak{l}_t)) = \mathfrak{l}$ for all t. Since there exists always at least on composition series through any given left ideal, we have $l(r(\mathfrak{l})) = \mathfrak{l}$ for any left ideal $\mathfrak{l}$. Similarly $r(l(\mathfrak{r})) = \mathfrak{r}$ for every right ideal $\mathfrak{r}$. That is, α) is always valid in A. This proves the first part of Theorem 6.

Our proof shows further that if $\mathfrak{l}/\mathfrak{l}' \cong \bar{A}\bar{e}_\kappa$ then $r(\mathfrak{l}')/r(\mathfrak{l}) \cong \bar{e}_{\pi(\kappa)}\bar{A}$. And this fact, together with the relation $d_l(\bar{A}\bar{e}_\kappa) = d_r(\bar{e}_\kappa\bar{A}) = f(\kappa)$, shows the validity of β') in case of a (not only quasi-Frobeniusean but) Frobeniusean ring A.

5. Corollaries

THEOREM 8. *Let A be a quasi-Frobeniusean ring. The composition length of a principal left ideal Ac is equal to that of the principal right ideal cA. If $c = c_1 + c_2$ and Ac is the direct sum $Ac = Ac_1 + Ac_2$, then cA is the direct sum $cA = c_1A + c_2A$. Further, if A is not only quasi-Frobeniusean but Frobeniusean, then $d_l(Ac) = d_r(cA)$.*

PROOF. Cf. I, Th. 4.

THEOREM 9. *Let A be quasi-Frobeniusean, and let $\pi(\kappa)$ have the same significance as in the definition. If $\mathfrak{l}'$ is a maximal left subideal of a left ideal $\mathfrak{l}$ in A and if the left module $\mathfrak{l}/\mathfrak{l}'$ is isomorphic to $\bar{A}\bar{e}_\kappa$, then the right module $r(\mathfrak{l}')/r(\mathfrak{l})$ is isomorphic to $\bar{e}_{\pi(\kappa)}\bar{A}$. The quasi-field of automorphisms of $\mathfrak{l}/\mathfrak{l}'$ is inversely isomorphic to that of $r(\mathfrak{l}')/r(\mathfrak{l})$. Furthermore, $r(N^\nu) = l(N^\nu)$ for any $\nu = 1, 2, \cdots$; where N is, as before, the radical of A.*

PROOF. The first assertion was shown in the proof of Th. 6. The second one is, by virtue of the first one, equivalent to the relation $\bar{e}_\kappa\bar{A}\bar{e}_\kappa \cong \bar{e}_{\pi(\kappa)}\bar{A}\bar{e}_{\pi(\kappa)}$, and this in turn can be seen in the same way as in I, Th. 3, vii). The final statement follows from the special case $r(N) = l(N)$, which was already established in the proof of Th. 6; See I, Th. 6.

THEOREM 10. *If in a ring A the relation α) holds for $\mathfrak{l} = \mathfrak{r} = N$ (radical), $\mathfrak{l} = \mathfrak{r} = 0$, and both α) and β') hold for all nilpotent simple two-sided ideals $\mathfrak{l} = \mathfrak{r} = \mathfrak{z}$, then A is Frobeniusean.*

PROOF. Arguments of i), ii), iii) and iv) of I, Th. 7 can be transferred term

by term. Now, $E_\kappa M (= E_\kappa M E_{\pi(\kappa)}) = \sum_{i=1}^{f(\kappa)} e_{\kappa,i}M$, and here the $f(\kappa)$ right ideals $e_{\kappa,i}M$ are mutually isomorphic and are direct sums of simple right ideals isomorphic to $\bar{e}_{\pi(\kappa)}\bar{A}$. Hence $d_r(E_\kappa M) \geqq f(\kappa)d_r(\bar{e}_{\pi(\kappa)}\bar{A}) = f(\kappa)f(\pi(\kappa))$ and the equality sign holds if and only if $e_{\kappa,i}M$ are simple. But $d_r(E_\kappa M) = d_l(A/l(E_\kappa M)) = d_l(A/N \cup A(E - E_\kappa)A)) = f(\kappa)^2$ by our assumption.[24] Thus $f(\kappa) \geqq f(\pi(\kappa))$. Since this is the case for every κ, necessarily $f(\kappa) = f(\pi(\kappa))$ and $e_{\kappa,i}M$ are simple. Similarly we find that $Me_{\kappa,i}$ are simple left ideals. It follows then easily that A is Frobeniusean.

THEOREM 11. *Suppose that* α) *holds for every nilpotent simple two-sided ideal* $\mathfrak{l} = \mathfrak{r} = \mathfrak{z}$ *as well as for* $\mathfrak{l} = \mathfrak{r} = N$, $\mathfrak{l} = \mathfrak{r} = 0$. *Let moreover*

$$\gamma) \qquad d_r(\mathfrak{z}) = d_r(A/r(\mathfrak{z})), \qquad d_l(\mathfrak{z}) = d_l(A/l(\mathfrak{z}))$$

for every nilpotent simple two-sided ideal $\mathfrak{z}$. *Then the ring* A *is Frobeniusean. Furthermore, the same is true when we replace the above* γ) *by*

$$\gamma') \qquad d_l(\mathfrak{z}) = d_l(A/r(\mathfrak{z})), \qquad d_r(\mathfrak{z}) = d_r(A/l(\mathfrak{z}))$$

or by

$$\gamma'') \qquad d_r(\mathfrak{z}) = d_l(A/r(\mathfrak{z})), \qquad d_l(\mathfrak{z}) = d_r(A/l(\mathfrak{z})).$$

PROOF. Arguments in i), ii), $\cdots$, iv) of I, Th. 7 remain again valid. And, again $d_r(E_\kappa M) \geqq f(\kappa)d_r(\bar{e}_{\pi(\kappa)}\bar{A}) = f(\kappa)f(\pi(\kappa))$; the equality sign is true if and only if $e_{\kappa,i}M$ are simple. Moreover, $d_r(A/r(E_\kappa M)) = d_r(A/N \cup A(E - E_{\pi(\kappa)})A) = f(\pi(\kappa))^e$, $d_r(A/l(E_\kappa M)) = d_r(A/N \cup A(E - E_\kappa)A) = f(\kappa)^2$ and $d_l(A/r(E_\kappa M)) = f(\pi(\kappa))^2$. Our assertions follow from these relations in quite a similar manner as above.

6. Vector moduli and a theorem of M. Hall

Let g be a natural number and consider a module $\mathfrak{A}$ consisting of all g-dimensional vectors $\mathfrak{x} = (x_1, x_2, \cdots, x_g)$ with components x_p from A. For a subset $\mathfrak{S}$ of A we denote by $R(\mathfrak{S})[L(\mathfrak{S})]$ the set of vectors $\mathfrak{y} = (y_1, y_2, \cdots, y_g)$ such that (the scalar product) $(\mathfrak{x}, \mathfrak{y}) = \sum_{p=1}^{g} x_p y_p[(\mathfrak{y}, \mathfrak{x}) = \sum_p y_p x_p] = 0$ for every $\mathfrak{x} \in \mathfrak{S}$. If we take our theorem 6 into account, the first part of the following theorem is equivalent to the principal theorem (Th. 5. 2) in Hall (7):

THEOREM 12. *Let* A *be quasi-Frobeniusean. Then* $L(R(\mathfrak{L})) = \mathfrak{L}$, $R(L(\mathfrak{R})) = \mathfrak{R}$ *for every* A*-left-submodule* $\mathfrak{L}$ *and* A*-right-submodule* $\mathfrak{R}$ *in* $\mathfrak{A}$. *If moreover* A *is Frobeniusean then* $d_l(\mathfrak{L}) = d_r(\mathfrak{A}/R(\mathfrak{L}))$ *and* $d_r(\mathfrak{R}) = d_l(\mathfrak{A}/L(\mathfrak{R}))$.

We want to show that we can derive this theorem also from our main theorems 6, 7, obtaining thus a second proof, though rather long, of Hall's theorem. For that purpose, consider the g-rowed matrix ring $B = A_g = \sum_{p,q=1}^{g} \epsilon_{pq} A$ over the quasi-Frobeniusean ring A; where ϵ_{pq} is a system of matrix units commutative with every element of A.

LEMMA 2. *B is quasi-Frobeniusean.*

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[24] Cf. I, Th. 7, ii).

PROOF. Evidently $\epsilon_{pp}e_{\kappa,i}$ $(p = 1, 2, \cdots, g; \kappa = 1, 2, \cdots, k; i = 1, 2, \cdots, f(\kappa))$ form a system of mutually orthogonal idempotent elements whose sum is the unit element of B. $Q = \sum \epsilon_{pq}N$ is the radical of B and $r(\sum \epsilon_{pq}N) = l(\sum \epsilon_{pq}N) = \sum \epsilon_{pq}M$; where N is, as before, the radical of A and M is the two-sided ideal $r(N) = l(N)$ in A. We put $P = \sum \epsilon_{pq}M$. From $e_\kappa M \cong e_{\pi(\kappa)}A/e_{\pi(\kappa)}N$ (with respect to A), it follows easily that $\epsilon_{pp}e_\kappa P \cong \epsilon_{pp}e_{\pi(\kappa)}B/\epsilon_{pp}e_{\pi(\kappa)}Q \cong \epsilon_{11}e_{\pi(\kappa)}B/\epsilon_{11}e_{\pi(\kappa)}Q$ with respect to B. Similarly $P\epsilon_{pp}e_{\pi(\kappa)} \cong B\epsilon_{11}e_\kappa/Q\epsilon_{11}e_\kappa$. But this shows that B is quasi-Frobeniusean.

Now, consider the A-left-module $\mathfrak{A}_1 = \epsilon_{11}B = \sum_p \epsilon_{1p}A$ and the A-right-module $\mathfrak{A}_2 = B\epsilon_{11} = \sum_p \epsilon_{p1}A$. The mapping θ_1: $\mathfrak{x} = (x_p) \rightarrow \sum \epsilon_{1p}x_p$ is an operator isomorphism of $\mathfrak{A}$ and $\mathfrak{A}_1$ with respect to the left operator ring A. Similarly, $\theta_2 : \mathfrak{y} = (y_p) \rightarrow \sum \epsilon_{1p}y_p$ is an operator isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}_2$ with respect to the right operator ring A. Moreover, $\theta_1(\mathfrak{x})\theta_2(\mathfrak{y}) = \sum \epsilon_{1p}x_py_p\epsilon_{p1} = (\mathfrak{x}, \mathfrak{y})\epsilon_{11}$. In particular $\theta_1(\mathfrak{x})\theta_2(\mathfrak{y}) = 0$ if and only if $(\mathfrak{x}, \mathfrak{y}) = 0$. Hence $\theta_2(R(\mathfrak{S}))(\mathfrak{S} \subseteq \mathfrak{A})$ is the set of right annihilators of $\theta_1(\mathfrak{S})$ in $\mathfrak{A}_2$, in the sense of multiplication in B. That is,

$$\theta_2(R(\mathfrak{S})) = \mathfrak{A}_2 \cap r(\theta_1(\mathfrak{S})). \tag{6}$$

Similarly

$$\theta_1(L(\mathfrak{S})) = \mathfrak{A}_1 \cap l(\theta_2(\mathfrak{S})). \tag{7}$$

On the other hand, $B(\mathfrak{l} \cap \mathfrak{A}_1) = \mathfrak{l}$ for every left ideal $\mathfrak{l}$ in B, while $B\mathfrak{L}_1 \cap \mathfrak{A}_1 = \mathfrak{L}_1$ for every (A-left-) submodule $\mathfrak{L}_1$ of $\mathfrak{A}$. (Because $\mathfrak{l} \cap \mathfrak{A}_1 = \epsilon_{11}\mathfrak{l}$ whence $B(\mathfrak{l} \cap \mathfrak{A}_1) = B\epsilon_{11}\mathfrak{l} = B\epsilon_{11}B\mathfrak{l} = B\mathfrak{l} = \mathfrak{l}$, and $B\mathfrak{L}_1 \cap \mathfrak{A}_1 = \epsilon_{11}B\mathfrak{L}_1 = \epsilon_{11}B\epsilon_{11}\mathfrak{L}_1 = A\epsilon_{11}\mathfrak{L}_1 = A\mathfrak{L}_1 = \mathfrak{L}_1$.) Similarly $(\mathfrak{r} \cap \mathfrak{A}_2)B = \mathfrak{r}$ and $\mathfrak{R}_2B \cap \mathfrak{A}_2 = \mathfrak{R}_2$ for every right ideal $\mathfrak{r}$ in B and (A-right-) submodule $\mathfrak{R}_2$ in $\mathfrak{A}_2$. We have now

$$\begin{aligned} \theta_1(L(R(\mathfrak{L}))) &= \mathfrak{A}_1 \cap l(\theta_2(R(\mathfrak{L}))) && \text{by (7)} \\ &= \mathfrak{A}_1 \cap l(\mathfrak{A}_2 \cap r(\theta_1(\mathfrak{L}))) && \text{by (6)} \\ &= \mathfrak{A}_1 \cap l((\mathfrak{A}_2 \cap r(B\theta_1(\mathfrak{L})))B) = \mathfrak{A}_1 \cap l(r(B\theta_1(\mathfrak{L}))) \\ &= \mathfrak{A}_1 \cap B\theta_1(\mathfrak{L}) && \text{(since } B \text{ is } \textit{quasi-Frobeniusean}) \\ &= \theta_1(\mathfrak{L}), \end{aligned}$$

whence $L(R(\mathfrak{L})) = \mathfrak{L}$. Similarly $R(L(\mathfrak{R})) = \mathfrak{R}$, and this proves the first part of the theorem. To prove the second part, one has only to notice that if A is Frobeniusean then B is so too, as can readily be seen, and that $d_l(B\mathfrak{L}_1)$ with respect to B is equal to $d_l(\mathfrak{L}_1)g$ with respect to A.

COROLLARY.[25] *Let $\mathfrak{G}$ be a finite group and let $\mathfrak{G}(A)$ be the group ring of $\mathfrak{G}$ over A. If A is quasi-Frobeniusean then $\mathfrak{G}(A)$ is so too.*

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[25] This corollary, as well as the above lemma 2, is closely related to the next section.

PROOF. Let $\mathfrak{G} = \{G_1, G_2, \cdots, G_g\}$. Let $\mathfrak{L}$ be a left ideal in $\mathfrak{G}(A)$. We want to show that an element $Y = \sum_p y_p G_p^{-1}$ in $\mathfrak{G}(A)$ is a right annihilator of $\mathfrak{L}$ if and only if $\sum x_p y_p = 0$ for all $X = \sum x_p G_p$ in $\mathfrak{L}$; if we succeed in showing this and the corresponding fact for a right ideal, then our corollary is an immediate consequence of Th. 6 and 12. Now, since $\sum x_p y_p$ is the coefficient of the group unit element in the product XY, the "only if" part is trivial. To prove the "if" part, suppose that Y satisfies the above condition. Let X be an arbitrary element in $\mathfrak{L}$. Then $G_p^{-1}X \in \mathfrak{L}$ for all $p = 1, 2, \cdots, g$. But the coefficient of G_p in XY is equal to the coefficient of the unit element in $G_p^{-1}XY$. It follows then that all the coefficients in XY vanish, that is, $XY = 0$. Hence $\mathfrak{L}Y = 0$.

Supplement for the case of an algebra. If A is a *Frobeniusean algebra* over a field F, then we can evidently replace d_l and d_r in the above theorem 12 by the dimension with respect to F. Furthermore, in this case we have the following generalization of Th. 2:

THEOREM 13. *Let A be a Frobeniusean algebra. Let $\mathfrak{L}$, $\mathfrak{L}_0$ be A-left-submoduli of the vector module $\mathfrak{A}$ such that $\mathfrak{L} \supseteqq \mathfrak{L}_0$, and denote the representations of A defined by the left module $\mathfrak{L}/\mathfrak{L}_0$ and the right module $R(\mathfrak{L}_0)/R(\mathfrak{L})$ by $a \to M(a)$ and $a \to N(a)$. Then the first representation $a \to M(a)$ is equivalent to the representation $a^* \to N(a)$, where $a \to a^*$ is the automorphism of A given in Theorem 1.*

PROOF. On retaining the notations of the section 1, we consider the linear function $\lambda((\mathfrak{x}, \mathfrak{y})) = \lambda(\sum x_p y_p)$ of the scalar product $(\mathfrak{x}, \mathfrak{y})$. And, for a subset $\mathfrak{S}$ in $\mathfrak{A}$ we denote by $\mathfrak{p}(\mathfrak{S})[\mathfrak{q}(\mathfrak{S})]$ the set of $\mathfrak{y}$ such that $\lambda((\mathfrak{x}, \mathfrak{y})) = 0$ $[\lambda((\mathfrak{y}, \mathfrak{x})) = 0]$ for all $\mathfrak{x} \in \mathfrak{S}$. It follows in quite a similar manner as before[26] that $\mathfrak{q}(\mathfrak{L}) = R(\mathfrak{L})$. And, the representation of A defined by the left module $\mathfrak{L}/\mathfrak{L}_0$ is equivalent to the one defined by the right module $\mathfrak{p}(\mathfrak{L}_0)/\mathfrak{p}(\mathfrak{L})$ (Cf. I, 8). Furthermore, φ: $\mathfrak{x} = (x_p) \to \mathfrak{x}^* = (x_p^*) = (\varphi(x_p))$ is a 1-1 mapping of $\mathfrak{A}$ on itself (See Th. 1), and it is characterized also by the relation $\lambda((\mathfrak{x}^*, \mathfrak{y})) = \lambda((\mathfrak{y}, \mathfrak{x}))$ (for all $\mathfrak{y}$). Evidently $\mathfrak{q}(\mathfrak{S})^* = \mathfrak{p}(\mathfrak{S})$, whence $R(\mathfrak{L})^* = \mathfrak{p}(\mathfrak{L})$. Thus the representation of A defined by $\mathfrak{L}/\mathfrak{L}_0$ is equivalent to the one defined by $R(\mathfrak{L}_0)^*/R(\mathfrak{L})^*$, and this is our assertion.

7. Product rings

Suppose that our operator domain Ω of A contains a field[27] F and that every operator in F commutes with every operator in Ω. Consider on the other hand a (finite linear) algebra B over F. By replacing in the usual manner the coefficient domain F by A we obtain a ring $C = B \times A$, which has Ω as a domain of operators.[28] (Namely, if $b_1, b_2, \cdots, b_n$ $(n = (B:F))$ is a basis of B, then

[26] Cf. I, 3.

[27] We naturally suppose that the addition and the multiplication in F coincide with those as operators and that the unit element in F is the identity operator.

[28] The conditions about the operator domain at the beginning of the section 4 are satisfied with regard to Ω and C. Moreover, C fulfills the minimum condition for right and left ideals.

$C = b_1A + b_2A + \cdots + b_nA$. To introduce the law of multiplication in C, we first define the product ba of $b = \sum \xi_i b_i \in B$, $a \in A$ to be $\sum_i b_i(\xi_i a)$ and then define $(\sum b_i a_i)(\sum b_i a_i')$ to be $\sum_{i,j}(b_i b_j)(a_i a_j')$. Further, $\alpha \sum b_i a_i = \sum b_i(\alpha a_i)$ for $\alpha \in \Omega$.)[29]

THEOREM 14. *The ring $C = B \times A$ is quasi-Frobeniusean [Frobeniusean] if and only if both A and B are so.*

PROOF. If $\mathfrak{l}_1$ and $\mathfrak{l}$ are left ideals in B and A respectively, the module product[30] $\mathfrak{l}_1\mathfrak{l}$ is evidently a left ideal in C, and

$$r(\mathfrak{l}_1\mathfrak{l}) = Br(\mathfrak{l}) \cup r(\mathfrak{l}_1)A \tag{8}$$

as can readily be seen.[31] (For a proof, take a basis $b_1, b_2, \cdots, b_n$ of B such that $r(\mathfrak{l}_1)$ is spanned by $b_{s+1}, b_{s+2}, \cdots, b_n$ ($s = n - (r(\mathfrak{l}_1):F)$). Suppose that an element $c = \sum b_i a_i$ in C is a right annihilator of the left ideal $\mathfrak{l}_1\mathfrak{l}$. Namely, if $b \in \mathfrak{l}_1$, $a \in \mathfrak{l}$ then $ba(\sum b_i a_i) = \sum (bb_i)(aa_i) = 0$. Let $bb_i = \sum_j \beta_{ij}(b)b_j$. Then $\sum_{i,j} b_j(\beta_{ij}(b)aa_i) = 0$, or

$$\sum_i \beta_{ij}(b)aa_i = 0 \qquad \text{for all } j = 1, 2, \cdots, n. \tag{9}$$

Here we notice that $\beta_{ij}(b) = 0$ if $i = s + 1, s + 2, \cdots, n$. Now, let b run over a system of generators $b^{(1)}, b^{(2)}, \cdots, b^{(t)}$ of $\mathfrak{l}_1$. Then the s matrices $B_i = (\beta_{ij}(b^{(p)}))_{jp}$ ($i = 1, 2, \cdots, s$) are linearly independent, because otherwise a non-trivial linear combination of $b_1, b_2, \cdots, b_s$ would be a right annihilator of $\mathfrak{l}_1$. In other words, the system of linear equations $\sum_{i=1}^{s} x_i\beta_{ij}(b^{(p)}) = 0$ ($j = 1, 2, \cdots, n$; $p = 1, 2, \cdots, t$) has no non-trivial solution, that is, the rank of the matrix $(\beta_{ij}(b^{(p)}))_{i,jp}$ ($i = 1, 2, \cdots, s$) is s. But then $\sum_{i=1}^{s} \beta_{ij}(b^{(p)})aa_i = 0$ (cf. (9))[32] implies $aa_i = 0$ ($i = 1, 2, \cdots, s$). Since a is any element in $\mathfrak{l}$, this shows that $a_1, a_2, \cdots, a_s \in r(\mathfrak{l})$ and $c = \sum b_i a_i = \sum_{i=1}^{s} b_i a_i + \sum_{i=s+1}^{n} b_i a_i \in Br(\mathfrak{l}) \cup r(\mathfrak{l}_1)A$. Thus $r(\mathfrak{l}_1\mathfrak{l}) \subseteq Br(\mathfrak{l}) \cup r(\mathfrak{l}_1)A$. The inclusion of the other direction is trivial.)

a) Now let A and B be quasi-Frobeniusean. Let N, Q be the radical of A, B respectively, and put $M = r(N) = l(N)$, $P = r(Q) = l(Q)$. The two-sided ideal $BN \cup QA$ in $C = B \times A$ is evidently contained in the radical S of C and we have, according to (8), $r(BN \cup QA) = r(BN) \cap r(QA) = BM \cap PA = PM$. Hence $r(S) \subseteq PM$.

Let g and e be any primitive idempotent elements in B and A respectively. There exist, since B and A are quasi-Frobeniusean, primitive idempotent elements g', e' in B, A such that $Pg \cong Bg'/Qg'$ ($\cong Bg' \cup Q/Q$) with respect to

[29] The structure of C thus obtained is of course independent of the special choice of the basis $b_1, b_2, \cdots, b_n$.

[30] The product of an element in B with an element in A is defined as in the above elucidation of $C = B \times A$.

[31] Here, as well as in the whole following proof, $r(*)$ denotes the set of right annihilators of the argument in A, B or C according as the argument lies in A, B or C.

[32] Cf. also a remark following (9).

(the left operator ring) B and $Me \cong Ae'/Ne'$ ($\cong Ae' \cup N/N$) with respect to A. Then, as can readily be seen,

$$(10) \quad PMge \cong Cg'e'/(BN \cup QA)g'e'(\cong Cg'e' \cup BN \cup QA/BN \cup QA)$$

with respect to C. Now the following fact is more or less well known and is easy to see:

LEMMA 3. *If in particular B and A are semisimple, then $B \times A$ is Frobeniusean*; (*it is in fact uni-serial* (= *einreihig*) *in the sense of Köthe* (9)[33]).

On applying the lemma to the simple algebra $Bg'B \cup Q/Q$ and the simple ring $Ae'A \cup N/N$ (instead of B, A), we find that $Cg'e'/Sg'e'$ is isomorphic, with respect to the left operator ring C, to the largest completely reducible submodule of $Cg'e'/(BN \cup QA)g'e'$. But this latter is, because of (10) and $r(S) \subseteq PM$, isomorphic to $r(S)ge$, and therefore

$$(11) \qquad Cg'e'/Sg'e' \cong r(S)ge.$$

Let $ge = j_1 + j_2 + \cdots + j_p$ and $g'e' = j_1' + j_2' + \cdots + j_q'$ are decompositions of ge and $g'e'$ into mutually orthogonal idempotent elements in C. Then the left side of (11) is a direct sum of exactly q simple submoduli, while the right side is a direct sum of at least p simple submoduli.[34] Hence $p \leqq q$. On the other hand, the right moduli $g'P$, $e'M$ of B, A are isomorphic to gB/gQ, eA/eN respectively and it follows in the same way as above that $q \leqq p$. Thus $q = p$, and all the left ideals $r(S)j_1$, $r(S)j_2$, $\cdots$, $r(S)j_p$ are necessarily simple. Moreover, they are isomorphic, up to their arrangement, with Cj_1'/Sj_1', Cj_2'/Sj_2', $\cdots$, Cj_q'/Sj_q'. Similarly the right ideals $j_1'l(S)$, $j_2'l(S)$, $\cdots$ are simple and isomorphic to j_1C/j_1S, j_2C/j_2S, $\cdots$ except their arrangement.

If we apply this consideration to all pairs of primitive idempotent elements g and e appearing in a decomposition of the unit elements in B and A, then we find readily that $C = B \times A$ is quasi-Frobeniusean, and that it is even Frobeniusean if both B and A are so.

(A second proof of the quasi-Frobeniusean part can be obtained in the following way: First let B be not only quasi-Frobeniusean but Frobeniusean. Then the parastrophic determinant of B is not identically zero, and an easy modification of the proof of the corollary in the preceeding section 6 shows that the relation α) holds in $C = B \times A$, whence C is quasi-Frobeniusean. The case of a quasi-Frobeniusean B can be reduced to this Frobeniusean case by a device similar to that of Part I, 3.)

b) Assume conversely that $C = B \times A$ is quasi-Frobeniusean. Then $r(C) = 0$, whence $r(B) = 0$ and $r(A) = 0$ because of (8). Hence the same relation (8) shows that $r(B\mathfrak{l}) = Br(\mathfrak{l})$ and $r(\mathfrak{l}_1 A) = r(\mathfrak{l}_1)A$. If we combine these relations and their (left, right) duals with the relation α) in C, which is valid by our

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[33] Cf. also the section 9 below.

[34] Observe that $r(S)j = r(S) \cap Cj$ is the largest completely reducible left subideal of Cj, which is not 0 evidently.

assumption, then we find that α) is valid also in both A and B. Hence A and B are quasi-Frobeniusean.

That if C is Frobeniusean then A and B are so can be seen from the analysis in the above a) which is, since A and B are quasi-Frobeniusean at least, now applicable.

(It is also possible to prove the first part of this b) by a structural analysis similar to a). On the other hand, in case (not only B but) A is an algebra over F, the second part of b) may be seen also by using (8) and the relation β) (I, Th. 1) or β') in C.)

We now prove the following supplement of Th. 12, Corollary:

Corollary. *Let A be Frobeniusean. Then the group ring $\mathfrak{G}(A)$ is Frobeniusean too.*

Proof. The A-left-moduli A/N and $M = r(N) = l(N)$ are isomorphic to each other. It follows easily that the $\mathfrak{G}(A)$-left-moduli $\mathfrak{G}(A/N)$ $(\cong \mathfrak{G}(A)/\mathfrak{G}(N))$ and $\mathfrak{G}(M)$ are isomorphic. In particular, the largest completely reducible $\mathfrak{G}(A)$-left-submodule of $\mathfrak{G}(M)$ is isomorphic to that of $\mathfrak{G}(A/N)$. But the latter is in turn isomorphic to the residue class module of $\mathfrak{G}(A/N)$ with respect to its radical, because the group ring $\mathfrak{G}(A/N)$ over the semisimple ring, A/N is Frobeniusean according to the above theorem 14.[35] Since $\mathfrak{G}(N)$ is evidently contained in the radical S of $\mathfrak{G}(A)$ and since $\mathfrak{G}(M)$ contains the right annihilator $r(S)$ of S in $\mathfrak{G}(A)$, we find that the residue class module $\mathfrak{G}(A)/S$ is isomorphic to the largest completely reducible left ideal of $\mathfrak{G}(A)$, with respect to the left operator domain $\mathfrak{G}(A)$. But we know already that $\mathfrak{G}(A)$ is quasi-Frobeniusean at least. And, these two facts assure that $\mathfrak{G}(A)$ is Frobeniusean.

8. Residue class rings

Theorem 15. *Let A be a Frobeniusean ring and let $\mathfrak{z}$ be a two sided ideal in A. The residue class ring $\tilde{A} = A/\mathfrak{z}$ is Frobeniusean if and only if the two-sided ideals $r(\mathfrak{z})$ and $l(\mathfrak{z})$ are respectively a principal right and a principal left ideal: $r(\mathfrak{z}) = bA$, $l(\mathfrak{z}) = Ac$.*

Proof. Let $\{\rho\}$ be the set of such indices that $E_\rho \notin \mathfrak{z}$. Since E_ρ are idempotent, $E_\rho \notin \mathfrak{z} \cup N$ either. (For, $(z + n)^r$ $(z \in \mathfrak{z}, n \in N)$ is a sum of n^r with an element in $\mathfrak{z}$, and n^r vanishes for a sufficiently high r.) Denote the residue classes of E_ρ, $e_{\rho,i}$ (mod $\mathfrak{z}$) by $\tilde{E}_\rho$, $\tilde{e}_{\rho,i}$. $\tilde{N} = N \cup \mathfrak{z}/\mathfrak{z}$ is the radical of $\tilde{A}$, and $\tilde{A}/\tilde{N} = \sum_\rho(\tilde{A}/\tilde{N})E_\rho$ is the (unique) decomposition of $\tilde{A}/\tilde{N}$ into a direct sum of simple two-sided ideals. $\tilde{e}_{\rho,i}$ are primitive idempotent elements in $\tilde{A}$, since $\tilde{A}\tilde{e}_{\rho,i}/N\tilde{e}_{\rho,i}$ are simple. Moreover, $E_\kappa(r(\mathfrak{z}) \cap M) \neq 0$ if and only if $\kappa \in \{\rho\}$, where $M = r(N) = l(N)$ as before, because $l(r(\mathfrak{z}) \cap M) = l(r(\mathfrak{z} \cup N)) = \mathfrak{z} \cup N$.

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[35] We decompose first A/N into a direct sum of simple rings and then consider the group rings over those simple components. They are Frobeniusean, because they can be considered as products of those simple rings with the group algebras constructed over their centers, for instance.

On the other hand, M is a direct sum of simple two-sided ideals $E_\kappa M = ME_{\pi(\kappa)}$ and every two-sided subideal of M is, as can readily be seen, a direct sum of some $E_\kappa M$. Hence

$$r(\mathfrak{z}) \cap M = \textstyle\sum_\rho E_\rho M (= \sum_\rho M E_{\pi(\rho)} = \sum_{\rho,i} M e_{\pi(\rho),i}). \tag{12}$$

a) Assume now $r(\mathfrak{z}) = bA$. We observe first that $e_{\kappa,i}b \neq 0$ if and only if $\kappa \in \{\rho\}$. The left ideal $Ae_{\rho,i}b$ is isomorphic to $A/l(e_{\rho,i}b)$ and here $l(e_{\rho,i}b) = l(e_{\rho,i}bA) = l(e_{\rho,i}A \cap bA)$[36] $= l(e_{\rho,i}A) \cup l(bA) = A(E - e_{\rho,i}) \cup \mathfrak{z}$. That is, $Ae_{\rho,i}b \cong A/A(E - e_{\rho,i}) \cup \mathfrak{z} \cong \bar{A}\bar{e}_{\rho,i}$. Moreover $Ab \cong A/l(b) = A/\mathfrak{z} = \bar{A}$. Since Ab is the sum of $Ae_{\rho,i}b$ while $\bar{A}$ is the direct sum of $\bar{A}\bar{e}_{\rho,i}$, we find that Ab is indeed the direct sum of $Ae_{\rho,i}b$.[37] Each $Ae_{\rho,i}b$ contains at least one simple left subideal, and therefore the largest completely reducible left subideal $Ab \cap M$ of Ab is a direct sum of at least $\sum_\rho f(\rho)$ simple left ideals. But $Ab \cap M \subseteq r(\mathfrak{z}) \cap M$, since $Ab \subseteq r(\mathfrak{z})$, and here $r(\mathfrak{z}) \cap M$ is the direct sum of $\sum_\rho f(\pi(\rho)) = \sum_\rho f(\rho)$ simple left ideals $Me_{\pi(\rho),i}$ (See (12)). Hence necessarily $Ab \cap M = r(\mathfrak{z}) \cap M$ and each $Ae_{\rho,i}b$ has only one simple left subideal. Furthermore, since $Ae_{\rho,1}b \cong \cdots \cong Ae_{\rho,f(\rho)}b$ (for $\bar{A}\bar{e}_{\rho,1} \cong \cdots \cong Ae_{\rho,f(\rho)}$), it follows that there exists a permutation $\{\nu(\rho)\}$ of $\{\rho\}$ such that the unique simple left subideal of $Ae_{\nu(\rho),i}b$ is, for each ρ, i, isomorphic to $Me_{\pi(\rho),i} \cong \bar{A}\bar{e}_\rho \cong \bar{A}\bar{e}_\rho/\bar{N}\bar{e}_\rho$. Thus $\bar{A}\bar{e}_{\nu(\rho),i}$ has also a unique simple left ideal, which is isomorphic to $\bar{A}\bar{e}_\rho/\bar{N}\bar{e}_\rho$. We observe also that $f(\nu(\rho)) = f(\pi(\rho)) = f(\rho)$ according to our construction.

Assume further $l(\mathfrak{z}) = Ac$. Then we find in the same way as above that every $\bar{e}_\rho\bar{A}$ has only one simple right subideal, and this simple right ideal is, by a remark in 4, necessarily isomorphic to $\bar{e}_{\nu(\rho)}\bar{A}/\bar{e}_{\nu(\rho)}\bar{N}$. $\bar{A}$ is therefore Frobeniusean.

b) To prove the converse, suppose that $\bar{A} = A/\mathfrak{z}$ is Frobeniusean. Let $\bar{P}$ be the annihilator ideal of the radical $\bar{N}$ in $\bar{A}$, and denote by P the two-sided ideal in A consisting of those elements whose residue classes (mod $\mathfrak{z}$) lie in $\bar{P}$. We consider further $r(\mathfrak{z})$ and $r(P)$ in A. The latter is the intersection of all maximal right subideals of the former, because P is the sum of all those left ideals in which $\mathfrak{z}$ is a maximal left subideal.

There exists, by definition, a permutation $\{\nu(\rho)\}$ of $\{\rho\}$ such that $\bar{e}_\rho\bar{P} \cong \bar{e}_{\nu(\rho)}\bar{A}$ $(\cong \bar{e}_{\nu(\rho)}\bar{A}/\bar{e}_{\nu(\rho)}\bar{N})$, $\bar{P}\bar{e}_{\nu(\rho)} \cong \bar{A}\bar{e}_\rho$ and $f(\nu(\rho)) = f(\rho)$. We have $\bar{E}_\rho\bar{P} = \bar{P}\bar{E}_{\nu(\rho)}$, or, $E_\rho P \cup \mathfrak{z} = PE_{\nu(\rho)} \cup \mathfrak{z}$. Now, $\mathfrak{z}$ coincides with the intersection $\bigcap_\rho(\sum_{\kappa\neq\rho} E_\kappa P \cup \mathfrak{z})$, ρ running over $\{\rho\}$. Hence

$$r(\mathfrak{z}) = \textstyle\sum_\rho r(\sum_{\kappa\neq\rho} E_\kappa P \cup \mathfrak{z}); \tag{13}$$

the summands are two-sided ideals since $\sum_{\kappa\neq\rho} E_\kappa P \cup \mathfrak{z}$ are such. Furthermore, $P/\sum_{\kappa\neq\rho} E_\kappa P \cup \mathfrak{z} \cong \bar{E}_\rho\bar{P}$ is, for each ρ, a direct sum of $f(\nu(\rho)) = f(\rho)$ simple A-left-submoduli isomorphic to $\bar{A}\bar{e}_\rho$, and therefore $r(\sum_{\kappa\neq\rho} E_\kappa P \cup \mathfrak{z})/r(P)$ is a

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[36] Observe that bA is two-sided.

[37] This follows also from Theorem 8.

direct sum of $f(\rho)$ simple A-right-submoduli isomorphic to $\bar{e}_{\pi(\rho)}\bar{A}$ (Cf. Theorem 9). But

$$(14)\quad \begin{aligned} r(\textstyle\sum_{\kappa\neq\rho}E_\kappa P \cup \mathfrak{z}) &= r(\textstyle\sum_{\kappa\neq\nu(\rho)}PE_\kappa \cup \mathfrak{z}) = r((P \cap A\textstyle\sum_{\kappa\neq\nu(\rho)}E_\kappa) \cup \mathfrak{z}) \\ &= (r(P) \cup E_{\nu(\rho)}A) \cap r(\mathfrak{z}) = r(P) \cup (E_{\nu(\rho)}A \cap r(\mathfrak{z}))^{38} = E_{\nu(\rho)}r(\mathfrak{z}) \cup r(P), \end{aligned}$$

whence

$$r(\textstyle\sum_{\kappa\neq\nu(\rho)}E_\kappa P \cap \mathfrak{z})/r(P) = E_{\nu(\rho)}r(\mathfrak{z}) \cup r(P)/r(P) = \textstyle\sum_{i=1}^{f(\nu(\rho))}(e_{\nu(\rho),i}r(\mathfrak{z}) \cup r(P)/r(P)).$$

Hence we find that the right moduli $e_{\nu(\rho),i}r(\mathfrak{z}) \cup r(P)/r(P)$ are simple and $\cong \bar{e}_{\pi(\rho)}\bar{A}$.

According to this fact, we now take for each ρ, $i(i = 1, 2, \cdots, f(\rho))$ an element $b_{\rho,i}$ in $e_{\nu(\rho),i}r(\mathfrak{z})e_{\pi(\rho),i}$ which is not contained in $r(P)$; observe that $f(\pi(\rho)) = f(\rho) = f(\nu(\rho))$. Then $b_{\rho,i}A \cup r(P) = e_{\nu(\rho),i}r(\mathfrak{z}) \cup r(P)$. Moreover, if we put $b = \sum_{\rho,i}b_{\rho,i}$, then $be_{\pi(\rho),i} = b_{\rho,i}$, and therefore

$$bA \cup r(P) = \textstyle\sum_{\rho,i}(e_{\nu(\rho),i}r(\mathfrak{z})) \cup r(P) = \textstyle\sum_\rho(E_{\nu(\rho)}r(\mathfrak{z})) \cup r(P) = r(\mathfrak{z})$$

(cf. (13), (14)). However, $r(P)$ is the intersection of all maximal right subideals of $r(\mathfrak{z})$, as we observed before. Hence necessarily $bA = r(\mathfrak{z})$.

Similarly $l(\mathfrak{z})$ is a principal left ideal, $l(\mathfrak{z}) = Ac$, and this completes the proof.

THEOREM 16. *Let a ring A satisfy (not only the minimum condition but) also the maximum condition for left and right ideals. In order that every residue class ring of A be Frobeniusean, it is necessary and sufficient that every two-sided ideal $\mathfrak{z}$ in A can be expressed as $\mathfrak{z} = Ac = cA$ $(c \in A)$.*

PROOF. a) Assume that every residue class ring of A is Frobeniusean. Then in particular A is Frobeniusean, and the above theorem 15 tells that every two-sided ideal $\mathfrak{z}(= r(l(\mathfrak{z}))$ is a principal right ideal $\mathfrak{z} = cA$. Moreover $d_r(\mathfrak{z}) = d_r(A) - d_r(A/\mathfrak{z}) = d_l(A) - d_l(A/\mathfrak{z}) = d_l(\mathfrak{z})$, for both A and $A/\mathfrak{z}$ are Frobeniusean.[39] But $d_r(\mathfrak{z}) = d_r(cA) = d_l(Ac)$ by Theorem 8. Hence $d_l(\mathfrak{z}) = d_l(Ac)$. Since $Ac \subseteq \mathfrak{z}$ we have $Ac = \mathfrak{z}$.

b) Assume next that the condition of the theorem is satisfied. The existence of a unit element in A can be seen in the same way as in I, Th. 10.[40] Let $\mathfrak{z} = Ac = cA$. The left ideal $\mathfrak{z} = Ac$ is isomorphic to $A/l(c) = A/l(cA) = A/l(\mathfrak{z})$, whence $d_l(\mathfrak{z}) = d_l(A/l(\mathfrak{z}))$. Since this is the case for every two-sided ideal, we have $r(l(\mathfrak{z})) = \mathfrak{z}$. For, $l(r(l(\mathfrak{z})) = l(\mathfrak{z})$ whence $d_l(r(l(\mathfrak{z})) = d_l(A/l(r(l(\mathfrak{z}))) = d_l(A/l(\mathfrak{z})) = d_l(\mathfrak{z})$, which, together with $r(l(\mathfrak{z})) \supseteq \mathfrak{z}$, implies $r(l(\mathfrak{z})) = \mathfrak{z}$. Similarly $d_r(\mathfrak{z}) = d_r(A/r(\mathfrak{z}))$ and $l(r(\mathfrak{z})) = \mathfrak{z}$. We find therefore, according to Theorem 11, that A is Frobeniusean. Our assertion that every residue class ring of A is Frobeniusean is now an immediate consequence of Theorem 15.

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[38] Modular law.

[39] See the footnote 21, or Th. 8.

[40] Here we use the composition lengths, and that is the reason that we assumed the maximum condition too.

9. Appendix: Generalized uni-serial rings

As a generalization of the notion of Köthe's *uni-serial* ring,[41] we introduce the following

Definition. We call A a *generalized uni-serial ring,* if A has a unit element and if every left ideal Ae as well as every right ideal eA generated by a primitive idempotent element e possesses only one composition series.[42]

The connection of this notion to our study is that a ring in which every residue class ring is Frobeniusean, as in the case of Theorem 16, is certainly a generalized uni-serial ring, although the converse is not true.

Now, consider a generalized uni-serial ring A. Let $N^{\rho-1} \neq 0$, $N^{\rho} = 0$, where N is, as before, the radical of A. Let $\sigma(\kappa)$ be, for each κ, an integer such that $e_\kappa N^{\sigma(\kappa)-1} \neq 0$, $e_\kappa N^{\sigma(\kappa)} = 0$; hence $\rho = \mathrm{Max}_\kappa(\sigma(\kappa))$. Since $e_\kappa A$ has only one composition series, every $e_\kappa N^i/e_\kappa N^{i+1}$ $(i = 0, 1, \cdots, \sigma(\kappa) - 1)$ is simple. We put

$$e_\kappa N^i/e_\kappa N^{i+1} \cong \bar{e}_{\varphi(\kappa,i)}\bar{A}.$$

Thus $e_\kappa N^i e_{\varphi(\kappa,i)} \not\subseteq N^{i+1}$ and if d is an element in $e_\kappa N^i e_{\varphi(\kappa,i)}$ not contained in N^{i+1} then $dA = e_\kappa N^i$. Let now $i + j \leqq \sigma(\kappa) - 1$. Then $e_\kappa N^{i+j} = e_\kappa N^i e_{\varphi(\kappa,i)} A N^j = e_\kappa N^i e_{\varphi(\kappa,i)} N^j$. Hence $j \leqq \sigma(\varphi(\kappa, i)) - 1$. Moreover, since $e_{\varphi(\kappa,i)} N^j = e_{\varphi(\kappa,i)} N^j e_{\varphi(\varphi(\kappa,i),j)} A$, we have $e_\kappa N^{i+j+1} \subset e_\kappa N^{i+j} = e_\kappa N^i e_{\varphi(\kappa,i)} N^j e_{\varphi(\varphi(\kappa,i),j)} A \subseteq e_\kappa N^{i+j} e_{\varphi(\varphi(\kappa,i),j)} A$. Therefore

$$\varphi(\kappa, i + j) = \varphi(\varphi(\kappa, i), j). \tag{15}$$

We put similarly $N^{\tau(\kappa)-1}e_\kappa \neq 0$, $N^{\tau(\kappa)}e_\kappa = 0$ and

$$N^i e_\kappa / N^{i+1} e_\kappa \cong \bar{A}\bar{e}_{\psi(\kappa,i)}$$

for $i \leqq \tau(\kappa) - 1$. Then

$$\psi(\kappa, i + j) = \psi(\psi(\kappa, i), j) \tag{16}$$

provided $i + j \leqq \tau(\kappa) - 1$. Further, if $i \leqq \tau(\kappa) - 1$ then $i \leqq \sigma(\psi(\kappa, i)) - 1$ and $\varphi(\psi(\kappa, i), i) = \kappa$, because $e_{\psi(\kappa,i)} N^i e_\kappa \not\subseteq N^{i+1}$. Hence, if $j \leqq \tau(\kappa) - 1$, $i \leqq \sigma(\psi(\kappa, j)) - 1$, then

$$\varphi(\psi(\kappa,j), i) = \begin{cases} \varphi(\varphi(\psi(\kappa, j), j), i - j) = \varphi(\kappa, i - j) \text{ or} \\ \varphi(\psi(\psi(\kappa, j - i), i), i) = \psi(\kappa, j - i) \end{cases} \tag{17}$$

according as $i \geqq j$ or $i \leqq j$.[43]

After this preparation we now proceed in a similar manner as in Köthe (9). Suppose that a left module $\mathfrak{M}$ of our A is a (restricted) direct sum $\sum_s \mathfrak{M}_s$ of (an arbitrary number of) submoduli $\mathfrak{M}_s$ and that each $\mathfrak{M}_s$ is homomorphic to

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[41] Köthe (9).

[42] See the footnotes 40 and 41 of Part I.

[43] We obtain a similar formula by interchanging σ, φ with τ, ψ. In particular, $\varphi(\psi(\kappa, j), i) = \psi(\varphi(\kappa, i), j)$ provided $j \leqq \tau(\kappa) - 1$, $i \leqq \sigma(\psi(\kappa, j)) - 1$, $i \leqq \sigma(\kappa) - 1$, $j \leqq \tau(\varphi(\kappa, i)) - 1$.

one of Ae_κ. Let $\mathfrak{M}_s$ be homomorphic to $Ae_{\kappa(s)}$ and let $e_{\kappa(s)}$ correspond to u_s in $\mathfrak{M}_s$; we have $\mathfrak{M}_s = Au_s = Ae_{\kappa(s)}u_s$. Denote further the composition length of $\mathfrak{M}_s$ by $i(s)$. Then $\mathfrak{M}_s \cong Ae_{\kappa(s)}/N^{i(s)}e_{\kappa(s)}$, and $N^{i(s)-1}u_s \neq 0$ while $N^{i(s)}u_s = 0$. Let m be the smallest of all $i(s)$.

Lemma 4. *If u ($\neq 0$) is an element in $e_\kappa\mathfrak{M}$ such that the composition length l of Au is not greater than m, then there exists an element v in $e_{\varphi(\kappa,m-l)}\mathfrak{M}$ such that $u \in e_\kappa N^{m-l}v$ and the composition length of Av is exactly m.*

Proof. Suppose for instance $u = a_1u_1 + a_2u_2 + \cdots + a_tu_t$, where $a_s \in Ae_{\kappa(s)}$ and $a_s \neq 0$ $(s = 1, 2, \cdots, t)$. Since $u = e_\kappa u = e_\kappa a_1u_1 + \cdots + e_\kappa a_tu_t$, we can, and shall, assume $a_s \in e_\kappa Ae_{\kappa(s)}$. Let $a_s \in e_\kappa N^{l(s)}e_{\kappa(s)}$ but $\notin e_\kappa N^{l(s)+1}e_{\kappa(s)}$. Then

$$\kappa = \psi(\kappa(s), l(s)), \qquad \kappa(s) = \varphi(\kappa, l(s)), \tag{18}$$

and $Aa_s = N^{l(s)}e_{\kappa(s)}$. On the other hand $N^lu = 0$ and $N^{l-1}u \neq 0$. From $N^ja_su_s = N^jAa_su_s = N^{j+l(s)}e_{\kappa(s)}u_s = N^{j+l(s)}u_s$ $(s = 1, 2, \cdots, t)$, it follows that $l + l(s) \geqq i(s)$ for all $s = 1, 2, \cdots, t$ while $l + l(s) \leqq$ (whence $=$) $i(s)$ for at least one s.

Take an element d such that $d \in e_\kappa N^{m-l}e_{\varphi(\kappa,m-l)}$ but $\notin N^{m-l+1}$. Then[44] $a_s \in e_\kappa N^{l(s)}e_{\kappa(s)} = e_\kappa N^{m-l}N^{l(s)-m+l}e_{\kappa(s)} = dN^{l(s)-m+l}e_{\kappa(s)}$, whence $a_s = db_s$ with $b_s \in e_{\varphi(\kappa,m-l)}N^{l(s)-m+l}e_{\kappa(s)}$. Here $\varphi(\varphi(\kappa, m - l), l(s) - m + l) = \varphi(\kappa, m - l + l(s) - m + l) = \varphi(\kappa, l(s)) = \kappa(s)$ according to (15), (18), and $Ab_s = N^{l(s)-m+l}e_{\kappa(s)}$.

Put now $v = b_1u_1 + b_2u_2 + \cdots + b_tu_t$. Then $u = dv$ and $e_{\varphi(\kappa,m-l)}v = v$. $N^mv = 0$, because $l + l(s) \geqq i(s)$ whence $N^mb_s \subseteq N^mN^{l(s)-m+l}e_{\kappa(s)} = N^{l(s)+l}e_{\kappa(s)} = 0$ $(s = 1, 2, \cdots, t)$. But $N^{m-1}v \neq 0$, since $N^{m-1}b_s = N^{l(s)+l-1}e_{\kappa(s)}$ and $l(s) + l \leqq i(s)$ for at least one s. Hence the composition length of Av is exactly m.

Lemma 5. *Let $\mathfrak{M}$ be the same as above. Suppose moreover that $\mathfrak{M}$ is contained in an over (A-left-) module $\mathfrak{m}$ and that $\mathfrak{m}$ is generated by $\mathfrak{M}$ and a cyclic module $\mathfrak{M}'$ which is homomorphic to one of Ae_κ and whose composition length does not exceed m. Then $\mathfrak{m}$ is a direct sum $\mathfrak{M} + \mathfrak{M}''$ of $\mathfrak{M}$ and a second module $\mathfrak{M}''$ homorphic to $\mathfrak{M}'$.*

Proof. Let $\mathfrak{M}' = Au \cong Ae_\kappa/N^he_\kappa$, $u = e_\kappa u$, and put $\mathfrak{U} = \mathfrak{M} \cap \mathfrak{M}'$. If $\mathfrak{U} = 0$, then we are already through. Suppose $\mathfrak{U} \neq 0$ and let $\mathfrak{U} = N^lu = N^le_\kappa u$; the composition length of $\mathfrak{U}$ is then $h - l$. Take an element c in $e_{\psi(\kappa,l)}N^le_\kappa$ not contained in N^{l+1}. Then $cu \in \mathfrak{U}$ and $Acu = N^lu = \mathfrak{U}$. But Acu is homomorphic to $Ae_{\psi(\kappa,l)}$. Hence there exists, by the above lemma 4, an element v in $\mathfrak{M}$ such that $v = e_\lambda v$, $Av \cong Ae_\lambda/N^me_\lambda$ and $cu \in e_{\psi(\kappa,l)}N^{m-(h-l)}v$, where $\lambda = \varphi(\psi(\kappa, l), m - (h - l)) = \varphi(\kappa, m - h)$.[45] But $e_{\psi(\kappa,l)}N^{m-(h-l)}e_\lambda = e_{\psi(\kappa,l)}N^lN^{m-h}e_\lambda = cN^{m-h}e_\lambda = ce_\kappa N^{m-h}e_\lambda$. Therefore $cu = cc'v$ with an element c' *in* $e_\kappa N^{m-h}e_\lambda$.

We put now $u'' = u - c'v$. Obviously $\mathfrak{m} = \mathfrak{M} \cup Au''$. Moreover, $e_\kappa u'' = e_\kappa u - e_\kappa c'v = u - c'v = u''$, whence Au'' is homomorphic to Ae_κ. Since $cu'' = cu - cc'v = 0$, we have $N^lu'' = Acu'' = 0$. Furthermore $\mathfrak{M} \cap Au'' = 0$, that is, $\mathfrak{m} = \mathfrak{M} + Au''$. For, if $au'' = w \in \mathfrak{M}(a \in Ae_\kappa)$, then $au - ac'v = w$, $au = w + ac'v \in \mathfrak{M}$. Hence $au \in \mathfrak{M} \cap Au = \mathfrak{U}$, and therefore $a \in N^le_\kappa$, $au'' = 0$. This completes our proof.

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[44] Observe that $l + l(s) \geqq i(s) \geqq m$.

[45] Cf. (17).

Now, the following theorem, which is an extension of the main theorem in Köthe (9), follows readily from the above lemma:

THEOREM 17.[46] *Let A be a generalized uni-serial ring. Then every left module of A is a (restricted) direct sum of cyclic submoduli, each of which is homomorphic to one of Ae_κ. The similar situation prevails for a right module of A.*

REMARK. In the special case where A is, as in Part I, an *algebra*, the converse is also true. Namely, an algebra which possesses the property of Theorem 17 is necessarily a generalized uni-serial algebra. For, if a left module $N^{i-1}e/N^ie$, where e is a primitive idempotent element, is not simple, then a right module which belongs to a representation of A equivalent to the one defined by the left module Ae/N^ie is homomorphic to none of $e_\kappa A$, whereas it is certainly directly indecomposable.[47]

Added in proof (Oct. 22, 1940): The remark adjoining the definition of Frobeniusean rings (Chapter II, 4) ought to have been placed after Theorem 6. In the proof of the theorem we showed $l(N) \subseteqq r(N)$ using only the condition i), and from this follows the relation $e_\kappa r(N) \supseteqq e_\kappa l(N) = \mathfrak{r}_\kappa$ used in the remark.

Asano (2), Theorem 2 and our Theorem 16 express essentially one and the same fact (Cf. Part I, Theorem 10 too). Namely, a ring whose residue class rings are all Frobeniusean is uni-serial. Further, the converse of Theorem 17 is valid generally. For all this cf. the writer's *Note on uni-serial and generalized uni-serial rings*, Proc. Imp. Acad. Tokyo Vol. XVI (1940) p. 285.

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[46] Cf. I, Th. 11.

[47] Cf. the proof of I, Th. 11 and I, footnote 40.

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ÜBER DIE TOPOLOGIE DER GRUPPEN-MANNIGFALTIGKEITEN UND IHRE VERALLGEMEINERUNGEN

BY HEINZ HOPF

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Einleitung.[1]

1. In der geschlossenen und orientierbaren Mannigfaltigkeit M sei eine "stetige Multiplikation" erklärt, das heißt: jedem geordneten Punktepaar (p, q) von M ist als "Produkt" ein Punkt pq von M zugeordnet, der stetig von dem Paar (p, q) abhängt. Setzen wir

$$pq = l_p(q),$$

so ist l_p bei festem p und variablem q eine Abbildung von M in sich; die Abbildungen l_p hängen stetig von dem Parameter p ab, und sie haben daher alle den gleichen Abbildungsgrad c_l. Analog ist der Grad c_r der Abbildungen r_q bestimmt, die durch

$$pq = r_q(p)$$

gegeben sind.

Definiert man etwa die stetige Multiplikation so, daß pq für alle (p, q) ein fester Punkt von M ist, so ist $c_l = c_r = 0$; setzt man $pq = p$ oder setzt man $pq = q$ für alle (p, q), so ist $c_l = 0$, $c_r = 1$ bzw. $c_l = 1$, $c_r = 0$. Diese trivialen stetigen Multiplikationen sind in jeder Mannigfaltigkeit möglich; dagegen kann man, wie sich zeigen wird, nur in sehr speziellen Mannigfaltigkeiten stetige Multiplikationen so definieren, daß

$$c_l \neq 0 \qquad \textit{und} \qquad c_r \neq 0$$

ist. Eine Mannigfaltigkeit,[2] welche eine solche Multiplikation zuläßt, soll eine Γ-*Mannigfaltigkeit* heißen.[2a]

Der Begriff der Γ-Mannigfaltigkeit ist eine Verallgemeinerung des Begriffes der *Gruppen*-Mannigfaltigkeit; ist nämlich M eine Gruppen-Mannigfaltigkeit, d.h. ist in M eine stetige Multiplikation erklärt, welche die Gruppen-Axiome erfüllt, so ist für den Punkt e, welcher die Gruppen-Eins darstellt, sowohl die

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* *Editors' Note.* This paper was originally submitted to *Compositio Mathematica*, August 23, 1939. It was transferred to the *Annals of Mathematics* (received November 18, 1940) after the *Compositio Mathematica* ceased publication.

[1] Eine kurze Ankündigung dieser Arbeit ohne Beweise ist in den C. R. 208 (1939), 1266–1267, erschienen.

[2] Unter einer "Mannigfaltigkeit" ist in dieser Arbeit immer eine *geschlossene und orientierbare* Mannigfaltigkeit zu verstehen.

[2a] Die Gültigkeit des assoziativen Gesetzes wird also nicht gefordert.

Abbildung l_e als auch die Abbildung r_e die Identität von M, und daher ist $c_l = c_r = 1$.

Somit gelten alle Sätze, die für Γ-Mannigfaltigkeiten bewiesen werden, insbesondere für geschlossenen Gruppen-Mannigfaltigkeiten.[3]

2. Wir werden Homologie-Eigenschaften von Mannigfaltigkeiten untersuchen; dabei soll *als Koeffizientenbereich der Körper der rationalen Zahlen* dienen.[4] Wie üblich fassen wir die Homologieklassen einer Mannigfaltigkeit M zu dem Homologie-Ring $\mathfrak{R}(M)$ zusammen: in ihm ist die Addition die der Bettischen Gruppen, und die Multiplikation ist durch die Schnitt-Bildung erklärt. Infolge der Benutzung rationaler Koeffizienten entgehen uns zwar gewisse Feinheiten der Struktur von M, so die etwa vorhandene Torsion; immerhin stimmen zwei Mannigfaltigkeiten M_1, M_2, deren rationalen Homologie-Ringe $\mathfrak{R}(M_1)$ und $\mathfrak{R}(M_2)$ einander dimensionstreu isomorph sind, in den wichtigsten algebraisch-topologischen Eigenschaften überein, insbesondere in den Werten der Bettischen Zahlen.

Unser Hauptziel ist der Beweis des folgenden Satzes:

Satz I. *Der Homologie-Ring $\mathfrak{R}(\Gamma)$ einer Γ-Mannigfaltigkeit Γ ist dimensionstreu isomorph dem Homologie-Ring $\mathfrak{R}(\Pi)$ eines topologischen Produktes*

$$\Pi = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_l}, \qquad l \geqq 1,$$

in welchem S_m die m-dimensionale Sphäre bezeichnet und alle Dimensionszahlen $m_1, m_2, \cdots, m_l$ ungerade sind.

3. Da man die Struktur der Ringe $\mathfrak{R}(\Pi)$ vollständig übersieht, kann man den Inhalt des Satzes I auch durch eine ausfürhliche Beschreibung der Struktur der Ringe $\mathfrak{R}(\Gamma)$ ausdrücken. Hierfür machen wir noch die folgenden terminologischen Bemerkungen:

Der Ring $\mathfrak{R}(M)$ einer beliebigen n-dimensionalen Mannigfaltigkeit[2] M enthält ein Eins-Element: es wird durch den orientierten n-dimensionalen Grundzyklus von M dargestellt; wir bezeichnen es durch 1. Die Dimension eines Elementes z von $\mathfrak{R}(M)$ nennen wir $d(z)$; daneben betrachten wir häufig die "duale Dimension" $\delta(z) = n - d(z)$. Unter einer "vollen additiven Basis" von $\mathfrak{R}(M)$ vorstehen wir die Vereinigung von Homologie-Basen der Dimensionen $0, 1, \cdots, n$.

Nun läßt sich der Satz I folgendermaßen aussprechen:

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[3] Die Topologie der Gruppen-Mannigfaltigkeiten wird in den folgenden beiden Schriften von E. Cartan behandelt: (a) *La Théorie des Groupes Finis et Continus et l'Analysis Situs* [Paris 1930, Mémorial Sc. Math. XLII]; (b) *La Topologie des Groupes de Lie* [Paris 1936, Actualités Scient. et Industr. **358**; sowie: L'Enseignement math. **35** (1936), 177-200; sowie: Selecta, Jubilé Scientifique, Paris 1939, 235-258].

[4] Tatsächlich werden wir von dem Koeffizientenbereich nur benutzen, daß er ein *Körper der Charakteristik* 0 ist.

Satz I (2. *Fassung*). *Aus dem Ringe $\mathfrak{R}(\Gamma)$ einer Γ-Mannigfaltigkeit Γ lassen sich Elemente $z_1, z_2, \cdots, z_l$ so auswählen, daß die 2^l Elemente*

$$1 \quad ; \quad z_i \quad ; \quad z_{i_1} \cdot z_{i_2} \quad (i_1 < i_2) \quad ;$$

$$z_{i_1} \cdot z_{i_2} \cdot z_{i_3} \quad (i_1 < i_2 < i_3) \quad ; \quad \cdots \quad ; \quad z_1 \cdot z_2 \cdot \cdots \cdot z_l$$

eine volle additive Basis bilden; die Multiplikation in $\mathfrak{R}(\Gamma)$ ist durch das distributive Gesetz, das assoziative Gesetz und die antikommutative Regel

$$z_j \cdot z_i = -z_i \cdot z_j$$

— in welcher speziell die Regel

$$z_i \cdot z_i = 0$$

enthalten ist—vollständig bestimmt; kurz: $\mathfrak{R}(\Gamma)$ ist der Ring der (inhomogenen) Multilinearformen in den antikommutativen Größen $z_1, z_2, \cdots, z_l$ mit rationalen Koeffizienten. Alle z_i sind homogendimensional,[5] *und ihre dualen Dimensionen $\delta(z_i) = m_i$ sind ungerade; die Dimension von Γ ist*

$$n = m_1 + m_2 + \cdots + m_l,$$

und allgemein ist

$$d(z_{i_1} \cdot z_{i_2} \cdot \cdots \cdot z_{i_r}) = n - m_{i_1} - m_{i_2} - \cdots - m_{i_r}.$$

Daß hierdurch gerade das Bestehen eines dimensionstreuen Isomorphismus zwischen $\mathfrak{R}(\Gamma)$ und $\mathfrak{R}(\Pi)$ ausgedrückt wird, erhellt aus der folgenden Tatsache, die man leicht bestätigt: bezeichnet man den orientierten Grundzyklus von S_m selbst mit S_m und mit p immer einen einfach gezählten Punkt, so besitzen in $R(\Pi)$ die Elemente

$$(*) \qquad \begin{aligned} Z_1 &= p \times S_{m_2} \times S_{m_3} \times \cdots \times S_{m_l}, \\ Z_2 &= S_{m_1} \times p \times S_{m_3} \times \cdots \times S_{m_l}, \\ &\cdots\cdots\cdots\cdots\cdots \\ Z_l &= S_{m_1} \times S_{m_2} \times S_{m_3} \times \cdots \times p \end{aligned}$$

genau die analogen Eigenschaften, welche wir in $\mathfrak{R}(\Gamma)$ soeben den Elementen $z_1, z_2, \cdots, z_l$ zugeschrieben haben. .

Aus der zweiten Fassung des Satzes I liest man unter anderem die folgende bemerkenswerte Eigenschaft der Γ-Mannigfaltigkeiten ab:

Satz Ib. *Jedes homogen-dimensionale Element z von $\mathfrak{R}(\Gamma)$, für welches die*

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[5] Ein Komplex K heißt homogen-dimensional von der Dimension r, wenn jedes Simplex von K auf einem r-dimensionalen Simplex von K liegt. Ein Element des Homologie-Ringes $\mathfrak{R}(M)$ heißt homogen-dimensional, wenn es in M durch einen homogen-dimensionalen Zyklus repräsentiert wird. Die additive Gruppe von $\mathfrak{R}(M)$ ist die direkte Summe der Gruppen der homogen r-dimensionalen Elemente mit $r = 0, 1, \cdots, n$.

duale Dimension $\delta(z)$ gerade und positiv ist, läßt sich durch Multiplikation und Addition aus höherdimensionalen Elementen erzeugen.

4. Der Satz I enthält weitgehende Aussagen über die Bettischen Zahlen einer Γ-Mannigfaltigkeit. Unter dem "Poincaréschen Polynom" eines Komplexes K verstehen wir das Polynom

$$P_K(t) = p_0 + p_1 t + p_2 t^2 + \cdots$$

in einer Unbestimmten t, wobei der Koeffizient p_r die r-te Bettische Zahl von K ist. Für die Sphäre S_m ist

$$P_{S_m}(t) = 1 + t^m;$$

bei der Bildung des topologischen Produktes $K_1 \times K_2$ zweier Komplexe K_1 und K_2 gilt nach der Formel von Künneth[6] die Regel

$$P_{K_1 \times K_2}(t) = P_{K_1}(t) \cdot P_{K_2}(t);$$

daher ist in dem Satz I (Nr. 2) der folgende Satz enthalten:

Satz I′. *Das Poincarésche Polynom einer Γ-Mannigfaltigkeit Γ hat die Gestalt*

$$P_\Gamma(t) = (1 + t^{m_1}) \cdot (1 + t^{m_2}) \cdot \cdots \cdot (1 + t^{m_l}), \tag{1}$$

wobei alle Exponenten m_i ungerade sind.

Wir heben einige der zahlreichen Beziehungen zwischen den Bettischen Zahlen hervor, die sich aus (1) ablesen lassen:

(a) *Die Eulersche Charakteristik ist* 0;[7]
denn die Charakteristik eines Komplexes K ist gleich $P_K(-1)$.

(b) *Die Summe der Bettischen Zahlen ist eine Potenz von* 2;[8]
denn diese Summe ist für einen Komplex K gleich $P_K(+1)$.

Γ sei n-dimensional; dann ist $p_n = 1$, $p_r = 0$ für $r > n$, also n der Grad von $P_\Gamma(t)$ und

$$m_1 + m_2 + \cdots + m_l = n.$$

Da der r-te Koeffizient des Polynoms (1) offenbar nicht größer ist als der r-te Koeffizient des Polynoms

$$(1 + t)^n = (1 + t)^{m_1} \cdot (1 + t)^{m_2} \cdots (1 + t)^{m_l},$$

so sieht man:

(c) *Es ist* $p_r \leqq \binom{n}{r}$ *für alle* r.[9]

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[6] Alexandroff-Hopf, Topologie I (Berlin 1935), 309, Formel (13′).

[7] Falls die Abbildungen l_p und r_q *topologisch* sind—also insbesondere, falls Γ eine *Gruppen*-Mannigfaltigkeit ist—, wird für $p_1 \neq p_2$ durch $f(q) = l_{p_1}^{-1} l_{p_2}(q)$ eine Abbildung von Γ auf sich erklärt, welche sich stetig in die Identität deformieren läßt und keinen Fixpunkt besitzt; dann folgt der obige Satz aus einem bekannten Fixpunktsatz.

[8] Für Gruppen-Mannigfaltigkeiten: Cartan[3] (b), 24.

[9] Für Gruppen-Mannigfaltigkeiten: H. Weyl, The classical groups [Princeton 1939], 279, als Korollar eines Satzes von Cartan (cf.[13]).

Man kann (1) in der Form

$$(1') \qquad P_\Gamma(t) = (1+t)^{l_1} \cdot (1+t^3)^{l_3} \cdot (1+t^5)^{l_5} \cdots, \qquad l_s \geqq 0,$$

schreiben; Ausrechnung ergibt

$$(1'') \qquad P_\Gamma(t) = 1 + l_1 t + \binom{l_1}{2} t^2 + \left(l_3 + \binom{l_1}{3}\right) t^3 + \left(l_1 l_3 + \binom{l_1}{4}\right) t^4 + \cdots$$

Es ist also

$$(2) \qquad p_1 = l_1 .$$

Da nun der Koeffizient von t^r in dem Produkt (1′) offenbar nicht kleiner ist als in dem Faktor

$$(1+t)^{l_1} = (1+t)^{p_1},$$

so gilt:

(d) *Es ist* $p_r \geqq \binom{p_1}{r}$ *für alle* r.[10]

Nach (1′) ist

$$l_1 + 3l_3 + 5l_5 + \cdots = n,$$

also nach (2):

$$p_1 = n - 3l_3 - 5l_5 - \cdots ;$$

daher läßt sich (c) für $r = 1$ verschärfen:

(e) *Es ist entweder* $p_1 = n$ *oder* $p_1 = n - 3$ *oder* $p_1 \leqq n - 5$.[11]

Ferner liest man aus (1″) und (2) die folgende Verschärfung von (d) für $r = 2$ ab:

(f) *Es ist* $p_2 = \binom{p_1}{2}$,

also speziell:

(f_0) *Ist* $p_1 = 0$ *oder* $p_1 = 1$, *so ist* $p_2 = 0$.[12]

Ebenso sieht man aus (1″) und (2):

(g) *Ist* $p_1 = 0$, *so ist auch* $p_4 = 0$.

Man kann ohne Mühe noch eine Reihe ähnlicher Relationen feststellen, z.B. die folgenden:

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[10] Für Gruppen-Mannigfaltigkeiten wie[9]; für wesentlich allgemeinere Räume mit stetiger Multiplikation: W. Hurewicz [Proc. Akad. Amsterdam **39** (1936), 215–224].

[11] Die Relation $p_1 \leqq n$ wurde für verallgemeinerte Gruppenräume zuerst von P. Smith [Annals of Math. (2) **36** (1935), 210–229] und dann von Hurewicz als Korollar aus dem unter[10] zitierten Satz bewiesen.

[12] Wegen der zweiten Bettischen Zahl einer Gruppen-Mannigfaltigkeit vergl. man Cartan[3], (b), 14 und 23–24.

(h) *Es sei* $p_1 = 0$; *dann ist*

$$3p_3 + 5p_5 + 7p_7 \leqq n,$$

$$p_{ir} \geqq \binom{p_i}{r} \textit{ für } i = 3, 5, 7 \textit{ und beliebiges } r.$$

5. Für *Liesche Gruppen* kann der Satz I, auf Grund von Sätzen, die wir E. Cartan und G. de Rham verdanken, aus der Sprache der Homologie-Theorie in die Sprache der Theorie ddr invarianten Differentialformen übersetzt werden.[13] Ich begnüge mich mit der Formulierung des Ergebnisses, im Anschluß an die 2. Fassung des Satzes I (Nr. 3):[14]

Die Mannigfaltigkeit G repräsentiere eine geschlossene Liesche Gruppe. Dann kann man aus der Gesamtheit der Differentialformen, welche in G invariant gegenüber den Operationen der Gruppe sind, Formen

$$\omega_1, \omega_2, \cdots, \omega_l,$$

deren Grade

$$m_1, m_2, \cdots, m_l$$

seien, so auswählen, daß sie die folgenden Eigenschaften besitzen:

1) *Für jedes r bilden diejenigen äußeren Produkte*

$$\omega_{i_1} \cdot \omega_{i_2} \cdot \cdots \cdot \omega_{i_\rho},$$

für welche

$$m_{i_1} + m_{i_2} + \cdots + m_{i_\rho} = r, \qquad i_1 < i_2 < \cdots < i_\rho$$

ist, eine lineare Basis (in Bezug auf konstante Koeffizienten) der invarianten Differentialformen des Grades r;

2) *alle* m_i *sind ungerade;*

3) *es ist* $m_1 + m_2 + \cdots + m_l$ *gleich der Dimension von G.*

Hierin ist unter anderem die folgende Tatsache enthalten, die dem Satz Ib (Nr. 3) entspricht:

Jede invariante homogene Differentialform geraden Grades läßt sich aus invarianten Differentialformen kleinerer Grade durch äußere Multiplikation und Addition erzeugen.

6. Auch bei Beschränkung auf Liesche Gruppen sind, soweit ich sehe, sowohl der Satz I als auch der schwächere Satz I′ neu. Allerdings waren diese Sätze bereits für eine so große und wichtige Reihe von Spezialfällen bekannt, daß ihre Gültigkeit für beliebige Liesche Gruppen vermutet werden konnte. L.

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[13] E. Cartan, Sur les invariants intégraux . . . [Annales Soc. polonaise de Math. **8** (1929), 181–225 (= Selecta, 203–233)]; G. de Rham, Sur l'analysis situs . . . [Journ. de Math. **10** (1931), 115–200].—Man vergl. auch H. Weyl, a.a.O.[9], 276 ff.

[14] Hier muß als Koeffizientenbereich der Körper der *reellen* Zahlen dienen; man vergl.[4]

Pontrjagin, R. Brauer und C. Ehresmann haben nämlich, mit verschiedenen Methoden, die Bettischen Zahlen derjenigen einfachen geschlossenen Lieschen Gruppen bestimmt, welche den vier großen Klassen in der Aufzählung von Killing-Cartan angehören, und diese Methoden liefern nicht nur den Satz I′, sondern auch den Satz I für die genannten Gruppen.[15]

Ausgehend von diesem Resultat könnte man wohl folgendermaßen zu einem Beweis des Satzes I für *alle* geschlossenen Lieschen Gruppen gelangen: Man verifiziere die Gültigkeit des Satzes auch an den fünf einfachen geschlossenen "Ausnahme"-Gruppen in der Killing-Cartanschen Aufzählung; dann übertrage man den—nunmehr für alle *einfachen* geschlossenen Lieschen Gruppen bewiesenen—Satz auf *alle* geschlossenen Lieschen Gruppen, indem man die, aus der Cartanschen Theorie bekannte, Rolle ausnützt, welche die einfachen Gruppen als Bausteine beliebiger Gruppen spielen.

Aber ganz abgesehen von der Frage, ob die direkte Bestätigung des Satzes an den fünf Ausnahme-Gruppen wirklich gelingt, würde ein solcher Beweis aus zwei Gründen nicht vollständig befriedigen. Erstens würde er so umfangreiche und tiefgehende Teile der Theorie der kontinuierlichen Gruppen als Hilfsmittel verwenden, daß dieser Aufwand in keinem rechten Verhältnis zu dem elementartopologischen Charakter des Satzes selbst stünde. Zweitens würde ein solcher Beweis in einer *Verifizierung* gipfeln; somit würde er zwar besonders konkrete Aufschlüsse über diejenigen *speziellen* Mannigfaltigkeiten liefern, an denen die Verifizierung stattfindet—also über die Mannigfaltigkeiten der einfachen geschlossenen Lieschen Gruppen—, es würde aber wohl doch der Wunsch nach einem Beweis offen bleiben, welcher *allgemeine* Gründe für die Gültigkeit des Satzes erkennen ließe.[16]

Daher glaube ich, daß selbst dann, wenn die direkte Verifizierung des Satzes I an den fünf Ausnahme-Gruppen und damit ein anderer Beweis für alle geschlossenen Lieschen Gruppen gelingt, doch unser Beweis, welcher für alle Γ-Mannigfaltigkeiten gilt und infolgedessen aus der Lieschen Theorie *nichts* benutzt, auch für die Lieschen Gruppen-Mannigfaltigkeiten willkommen ist.

7. Andererseits weiß man, daß der Satz I gewisse Verschärfungen erlaubt, wenn man sich auf Gruppen-Mannigfaltigkeiten beschränkt; dann unterliegen nämlich die Zahlen m_i, die im Satz I auftreten, gewissen Gesetzen; so folgt aus Sätzen von E. Cartan:[17] entweder sind alle $m_i = 1$ — dann ist die Gruppe Abelsch—, oder wenigstens ein m_i ist gleich 3. Diesen Satz oder ähnliche Sätze

[15] L. Pontrjagin [C. R. Acad. Sc. U.R.S.S. **1** (1935), 433–437 und C. R. Paris, **200** (1935), 1277–1280].—R. Brauer [C. R. **201** (1935), 419–421] (man vergl. auch H. Weyl, a.a.O.[9], 232 ff.).—C. Ehresmann [C. R. **208** (1939), 321–323; 1263–1265].

[16] Cartan[3], (b), 26: "... Mais même en nous bornant à la simple détermination des nombres de Betti des groupes simples, on ne devra pas s'estimer complètement satisfait si on arrive à faire cette détermination pour les cinq groupes exceptionnels.... Il faut espérer qu'on trouvera aussi une raison de portée générale expliquant la forme si particulière des polynomes de Poincaré des groupes simples clos."

[17] A.a.O.[3]: (a), 42–43; (b), 24.

mit unserer Methode zu beweisen, welche immer alle Γ-Mannigfaltigkeiten gleichzeitig behandelt, ist prinzipiell unmöglich; denn für Γ-Mannigfaltigkeiten unterliegen die m_i überhaupt keiner Einschränkung; es gilt nämlich der folgende Satz:

SATZ II. *Jedes Sphären-Produkt*

$$S_{m_1} \times S_{m_2} \times \cdots \times S_{m_l}, \qquad l \geqq 1,$$

in welchem die Dimensionszahlen $m_1, m_2, \cdots, m_l$ *ungerade sind, ist eine* Γ-*Mannigfaltigkeit.*

Aus diesem Satz geht hervor, daß der Begriff der Γ-Mannigfaltigkeit nicht nur seiner Definition nach, sondern auch tatsächlich viel allgemeiner ist als der Begriff der Gruppen-Mannigfaltigkeit: nach Satz II sind alle Sphären S_{2k+1} Γ-Mannigfaltigkeiten, während nach einem bekannten, soeben erwähnten Satz von Cartan unter allen Sphären S_n allein S_1 und S_3 Gruppenräume sind.

8. Die Aufgabe, diejenigen Ringe aufzuzählen, welche als Homologie-Ringe von Γ-Mannigfaltigkeiten auftreten, ist durch die Sätze I und II vollständig gelöst.

Die Gültigkeit des Satzes II wird rasch im §1 durch direkte Angabe geeigneter stetiger Multiplikationen bestätigt.

Im §2 werden Erzeugenden-Systeme beliebiger Homologie-Ringe betrachtet. Im Rahmen dieser Betrachtung wird der Satz I in zwei Teile zerlegt—Satz Ia und Satz Ib, von denen wir den zweiten schon in Nr. 3 ausgesprochen haben. Im Satz Ib (Nr. 15) tritt der Begriff des "maximalen" Elementes eines Homologie-Ringes auf, der auch für andere Zwecke als unseren gegenwärtigen wichtig und brauchbar sein dürfte; wir werden sogleich noch auf ihn zurückkommen (Nr. 9).

Der Ansatz zum Beweis der Sätze Ia und Ib, und damit des Satzes I, ist der folgende: man fasse die Punktepaare (p, q) von M als die Punkte $p \times q$ der Produkt-Mannigfaltigkeit $M \times M$ auf; durch eine stetige Multiplikation pq in M, wie wir sie in Nr. 1 erklärt haben, ist dann eine stetige Abbildung F von $M \times M$ in M bestimmt: $F(p \times q) = pq$; diese Abbildungen F sind mit Hilfe des "Umkehrungs-Homomorphismus" zu untersuchen. Entsprechend diesem Ansatz werden zunächst im §3 einige einfache Eigenschaften des Ringes $\mathfrak{R}(M \times M)$ zusammengestellt; sodann wird im §4, nachdem an seinem Anfang kurz an die Theorie des Umkehrungs-Homomorphismus erinnert worden ist, der Beweis der Sätze Ia und Ib geführt.

9. Der schon erwähnte Begriff des maximalen Elementes ist der folgende: ein homogen-dimensionales Element eines Homologie-Ringes $\mathfrak{R}(M)$ heißt maximal, wenn es nicht durch Multiplikation und Addition aus höherdimensionalen Elementen erzeugt werden kann. Im §5 werden die maximalen Elemente noch etwas näher betrachtet, und es werden ihnen jetzt die "minimalen" Elemente gegenübergestellt: das sind diejenigen homogen-dimensionalen Ele-

mente v von $\mathfrak{R}(M)$, welche keine Vielfachen $w = u \cdot v$ mit $0 < d(w) < d(v)$ besitzen. Die Untersuchung führt erstens leicht zu einem Satz über eine gewisse Dualität zwischen den maximalen und den minimalen Elementen (Nr. 33) und zweitens, unter Benutzung des Umkehrungs-Homomorphismus, zu einer bemerkenswerten Invarianz-Eigenschaft der minimalen Elemente (Nr. 34). Diese Tatsachen, zusammen mit dem Satz Ib, liefern noch als Korollar den folgenden Satz, der eine kräftige Verallgemeinerung der Tatsache darstellt, daß eine Sphäre gerader Dimension nicht als Gruppen-Mannigfaltigkeit auftreten kann:

Satz III. *In den Γ-Mannigfaltigkeiten sind die stetigen Bilder von Sphären gerader Dimension immer homolog* 0.

Zum Schluß (Nr. 37) wird ein Problem formuliert, das durch die erwähnte Methode von Pontrjagin[15] angeregt ist und das für die weitere topologische Untersuchung der Gruppen-Mannigfaltigkeiten wichtig sein dürfte; es wird eine Vermutung ausgesprochen, in welcher die minimalen Elemente einer Gruppen-Mannigfaltigkeit eine Hauptrolle spielen.

§1. Beweis des Satzes II.

Der Satz II (Nr. 7) läßt sich in die folgenden beiden Teile zerlegen:

Satz IIa. *Für ungerades m ist die Sphäre S_m eine Γ-Mannigfaltigkeit.*[18]

Satz IIb. *Das topologische Produkt $\Gamma \times \Gamma'$ zweier Γ-Mannigfaltigkeiten Γ und Γ' ist selbst eine Γ-Mannigfaltigkeit.*

10. *Beweis des Saltzes* IIa.[19] Für jeden Punkt q der Sphäre S_m bezeichne r_q die Spiegelung der S_m an demjenigen Durchmesser, auf welchem q liegt; wir setzen $pq = l_p(q) = r_q(p)$.

Die Abbildung r_q ist topologisch, also ist $c_r = \pm 1$ (und zwar, wie man leicht sieht, $c_r = -1$). Wir behaupten weiter: $c_l = \pm 2$ (und zwar ist $c_l = +2$).

p sei ein fester Punkt auf S_m; durch l_p wird jeder Großkreis, auf dem p liegt, auf sich abgebildet, und zwar folgendermaßen: führt man auf dem Kreis eine Winkelkoordinate mit p als Nullpunkt ein, und ist dann q der Punkt mit der Koordinate α, so hat $pq = l_p(q)$ die Koordinate 2α. Daraus ergibt sich: sowohl die offene Halbkugel H von S_m, deren Mittelpunkt p ist, als auch ihre antipodische Halbkugel H' wird topologisch auf $S_m - p'$ abgebildet, wobei p' der Antipode von p ist; die gemeinsame Randsphäre von H und H' geht in den Punkt p' über. Bezeichnen wir mit q' immer den Antipoden von q, so ist der Zusammenhang swischen den Abbildungen der beiden Halbkugeln H und H' durch die Beziehung $l_p(q') = l_p(q)$ gegeben. Nun hat bei ungeradem m die Involution der S_m, welche je zwei Antipoden vertauscht, den Grad $+1$; daher haben, wenn wir die Orientierungen von H und H' durch eine feste Orientierung der S_m festlegen, die topologischen Abbildungen l_p von H und H' den *gleichen*

[18] Daß für *gerades* m die S_m *nicht* Γ-Mannigfaltigkeit ist, ist in Nr. 4 (a) und in Satz III (Nr. 9) enthalten.

[19] Wiedergabe des Beweises von "Satz IV" aus meiner Arbeit in den Fund. Math. **25** (1935), 427–440.

Grad $\epsilon = \pm 1$ (und zwar, wie man leicht sieht, $+1$). Daher hat die Abbildung l_p der ganzen S_m auf sich den Grad $2\epsilon = \pm 2$ (und zwar $+2$).

11. *Beweis des Satzes* IIb. Die stetigen Multiplikationen in Γ und Γ' seien durch

$$pq = l_p(q) = r_q(p) \qquad \text{bzw,} \qquad p'q' = l'_{p'}(q') = r'_{q'}(p')$$

gegeben; die zugehörigen Grade seien c_l, c_r, $c_{l'}$, $c_{r'}$; sie sind sämtlich $\neq 0$. Wir definieren in der Mannigfaltigkeit $\Gamma \times \Gamma'$, deren Punkte mit $p \times p'$, $q \times q'$, $\cdots$ bezeichnet werden, eine stetige Multiplikation durch die Festsetzung

$$(p \times p')\cdot(q \times q') = pq \times p'q' = L_{p\times p'}(q \times q') = R_{q\times q'}(p \times p');$$

die zugehörigen Grade seien C_L, C_R. Der Satz ist bewiesen, sobald gezeigt ist:

$$C_L = c_l \cdot c_{l'}, \qquad C_R = c_r \cdot c_{r'}.$$

Die Gültigkeit dieser Gleichheiten ist in dem folgenden *Hilfssatz* enthalten:

f und f' seien Abbildungen[20] der Mannigfaltigkeiten[2] A und A' in die Mannigfaltigkeiten B bzw. B', welche die gleichen Dimensionen haben wie A bzw. A'; die Grade von f und f' seien c bzw. c'. Dann hat die Abbildung F von $A \times A'$ in $B \times B'$, die durch

$$F(p \times p') = f(p) \times f'(p')$$

gegeben ist, wobei p, p' die Punkte von A bzw. A' durchlaufen, den Grad cc'.

Für den Beweis ersetzen wir f und f' durch so gute simpliziale Approximationen f_1, f'_1, daß auch diese die Grade c, c' haben, und daß auch die Abbildung F_1 von $A \times A'$ in $B \times B'$, die durch

$$F_1(p \times p') = f_1(p) \times f'_1(p')$$

gegeben ist, den gleichen Grad C hat wie F. Die Grundsimplexe der Zerlegungen von A, A', B, B', welche den simplizialen Abbildungen f_1, f'_1 zugrunde liegen, seien mit u_i, u'_j, v_k, v'_l bezeichnet; dann bilden die Produkte $u_i \times u'_j$ und $v_k \times v'_l$ die Grundzellen von Zellenzerlegungen der Mannigfaltigkeiten $A \times A'$ bzw. $B \times B'$. Durch F_1 wird jede Zelle $u_i \times u_j$ affin abgebildet, und zwar folgendermaßen: ist

$$f_1(u_i) = 0 \qquad \text{oder} \qquad f'_1(u'_j) = 0,$$

wird also die Dimension wenigstens eines der Simplexe u_i, u'_j durch die Abbildung f_1 oder f'_1 erniedrigt, so wird auch die Dimension der Zelle $u_i \times u'_j$ durch die Abbildung F_1 erniedrigt, es ist also $F_1(u_i \times u'_j) = 0$; ist

$$f_1(u_i) = \epsilon v_k, \qquad f'_1(u'_j) = \epsilon' v'_l, \qquad \epsilon = \pm 1, \qquad \epsilon' = \pm 1,$$

[20] Alle vorkommenden "Abbildungen" von Mannigfaltigkeiten sollen *eindeutig und stetig* sein.

sind also die beiden Abbildungen f_1, f_1' nicht-singulär, so ist auch die Abbildung F_1 von $u_i \times u_j'$ nicht-singulär, und es ist, wie sich aus bekannten Vorzeichenregeln bei der Bildung topologischer Produkte ergibt,

$$F_1(u_i \times u_j') = \epsilon\epsilon'(v_k \times v_l').$$

Jetzt lehrt eine leichte Abzählung: die algebraische Bedeckungszahl—d.h. die Anzahl der positiven Bedeckungen, vermindert um die Anzahl der negativen Bedeckungen—einer festen Grundzelle von $B \times B'$, etwa der Zelle $v_k \times v_l'$, also der Grad C von F_1, ist gleich dem Produkt der algebraischen Bedeckungszahlen von v_k und v_l' bei den Abbildungen f_1 bzw. f_1', also gleich cc'.

§2. Irreduzible Erzeugenden-Systeme und maximale Elemente eines Homologie-Ringes. Umformung des Satzes I.

12. *Vorbemerkungen.* Es sei M eine n-dimensionale Mannigfaltigkeit. Wie schon in Nr. 3 festgesetzt, bezeichnen wir die Dimension eines Elementes z von $\mathfrak{R}(M)$ mit $d(z)$ und verstehen unter seiner dualen Dimension die Zahl $\delta(z) = n - d(z)$.

Bekanntlich ist für homogen-dimensionale z, z' auch $z \cdot z'$ homogen-dimensional und

$$\delta(z \cdot z') = \delta(z) + \delta(z')^{21}) \tag{2.1}$$

sowie

$$z' \cdot z = \pm z \cdot z', \tag{2.2}$$

und zwar[22])

$$z' \cdot z = (-1)^{\delta(z) \cdot \delta(z')} z \cdot z', \tag{2.3}$$

also speziell

$$z \cdot z = 0 \qquad \text{bei ungeradem } \delta(z). \tag{2.4}$$

13. *Erzeugenden-Systeme.* Die homogen-n-dimensionalen Elemente von $\mathfrak{R}(M)$, also die rationalen Vielfachen der Eins des Ringes, nennen wir die "skalaren" Elemente von $\mathfrak{R}(M)$.

Unter einem "Erzeugenden-System" von $\mathfrak{R}(M)$ verstehen wir ein solches System von homogen-dimensionalen, nicht-skalaren Elementen $z_1, z_2, \cdots, z_L$, daß man alle erhält, wenn man auf $z_1, z_2, \cdots, z_L$ und 1 die Operationen der gegenseitigen Multiplikation, der Multiplikation mit rationalen Koeffizienten und der Addition ausübt.

Auf Grund der Regel (2.2) kann man jedes Element von $\mathfrak{R}(M)$ auf wenigstens eine Weise als Polynom in den z_λ, d.h. als Summe von Ausdrücken

$$t \cdot z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdot \cdots \cdot z_L^{\alpha_L}, \qquad a_\lambda \geqq 0, \tag{2.5}$$

mit rationalen Koeffizienten t schreiben.

[21] Dem Null-Element des Ringes $\mathfrak{R}(M)$ wird *jede* Dimensionszahl zugeschrieben.
[22] Man vergl. z.B. Lefschetz, *Topology* [New York 1930], 166.

Ein Erzeugenden-System heißt "irreduzibel", wenn keines seiner echten Teilsysteme bereits ein Erzeugenden-System ist.

Offenbar ist in jedem Erzeugenden-System wenigstens ein irreduzibles Erzeugenden-System enthalten.

Man zeigt übrigens leicht, daß die Anzahl l der Elemente eines irreduziblen Erzeugenden-Systems von $\mathfrak{R}(M)$ nicht von diesem speziellen System abhängt, sondern eine Invariante von M ist; wir werden diese Tatsache, die wir vorläufig nicht benutzen, später beweisen (Nr. 31).

14. *Maximale Elemente.* Ein Element z von $\mathfrak{R}(M)$ heißt "maximal", wenn es 1) homogen-dimensional und nicht-skalar ist, und wenn es 2) nicht in dem Teilring von $\mathfrak{R}(M)$ enthalten ist, der von den homogen-dimensionalen Elementen z' von $\mathfrak{R}(M)$ mit $d(z') > d(z)$ erzeugt wird.[23]

Wir behaupten: *Jedes Element z_λ eines irreduziblen Erzeugenden-Systems $(z_1, z_2, \cdots, z_l)$ ist maximal.*

Beweis: Daß z_λ homogen-dimensional und nicht-skalar ist, ist in der Definition des Erzeugenden-Systems enthalten. Wäre z_λ nicht maximal, so wäre z_λ Element des Ringes $\mathfrak{U}$, der von allen homogen-dimensionalen Elementen z' mit $d(z') > d(z_\lambda)$ erzeugt wird. Nun läßt sich aber jedes dieser z' als Polynom in den Erzeugenden $z_1, z_2, \cdots, z_l$ schreiben, und hierbei tritt aus Dimensionsgründen das Element z_λ nicht auf; aus $z_\lambda \in \mathfrak{U}$ würde daher folgen, daß auch z_λ selbst ein Polynom in den von z_λ verschiedenen Elementen des Systems $(z_1, z_2, \cdots, z_l)$ wäre; dann würde aber dieses System, wenn man aus ihm z_λ wegließe, immer noch ein Erzeugenden-System bleiben—entgegen seiner Irreduzibilitäts-Eigenschaft.

15. *Umformung des Satzes I.*

Satz Ia. *$(z_1, z_2, \cdots, z_l)$ sei ein irreduzibles Erzeugenden-System des Ringes $\mathfrak{R}(\Gamma)$ einer Γ-Mannigfaltigkeit. Dann ist*

$$z_1 \cdot z_2 \cdot \cdots \cdot z_l \neq 0. \tag{2.6}$$

Satz Ib. *z sei ein maximales Element des Ringes $\mathfrak{R}(\Gamma)$ einer Γ-Mannigfaltigkeit. Dann ist $\delta(z)$ ungerade.*

Wir werden diese beiden Sätze im §4 beweisen. Jetzt wollen wir nur zeigen, daß aus ihnen der Satz I (Nr. 2, 3) folgt; dies wird geschehen sein, sobald wir bewiesen haben:

Es sei $(z_1, z_2, \cdots, z_l)$ ein irreduzibles Erzeugenden-System von $\mathfrak{R}(\Gamma)$, und es sei bekannt, daß die Sätze Ia und Ib gelten; dann haben $z_1, z_2, \cdots, z_l$ die in Nr. 3 genannten Eigenschaften.

Zunächst ergibt sich aus Nr. 14, daß alle Elemente z_i maximal, also aus Satz Ib, daß alle Zahlen

$$\delta(z_i) = m_i, \qquad i = 1, 2, \cdots, l, \tag{2.7}$$

[23] Insbesondere ist jedes homogen $(n-1)$-dimensionale Element, das $\neq 0$ ist, maximal.

ungerade sind. Nach (2.3) ist daher

$$z_j \cdot z_i = -z_i \cdot z_j, \tag{2.8}$$

also speziell

$$z_i \cdot z_i = 0. \tag{2.9}$$

Da die z_i ein Erzeugenden-System bilden, kann man jedes Element von $\mathfrak{R}(\Gamma)$ als Summe von Ausdrücken (2.5)—mit $L = l$—darstellen; infolge von (2.9) kann man sich dabei aber auf die Exponenten $\alpha_\lambda = 0$ und $\alpha_\lambda = 1$ beschränken; das heißt: jedes Element z von $\mathfrak{R}(\Gamma)$ läßt sich auf wenigstens eine Weise als lineare Verbindung mit rationalen Koeffizienten der Elemente

$$1; \quad z_i; \quad z_{i_1} \cdot z_{i_2} \; (i_2 < i_2); \quad z_{i_1} \cdot z_{i_2} \cdot z_{i_3} \; (i_1 < i_2 < i_3); \cdots ; \quad z_1 \cdot z_2 \cdot \cdots \cdot z_l \tag{2.10}$$

darstellen, also in der Form

$$z = t + \sum t_i z_i + \sum t_{i_1 i_2} z_{i_1} \cdot z_{i_2} + \sum t_{i_1 i_2 i_3} z_{i_1} \cdot z_{i_2} \cdot z_{i_3} + \\ + \cdots + t_{12 \ldots l} z_1 \cdot z_2 \cdot \cdots \cdot z_l, \tag{2.11}$$

wobei die Koeffizienten $t, t_i, t_{i_1 i_2}, \cdots$ rational sind und die Indices die unter (2.10) angedeuteten Bedingungen erfüllen. Wir haben zu zeigen, daß die Elemente (2.10) eine additive Basis bilden, d.h. daß sie linear unabhängig sind (in Bezug auf rationale Koeffizienten), mit anderen Worten: in einer Darstellung (2.11) des Elementes $z = 0$ verschwinden alle Koeffizienten auf der rechten Seite.

Es sei also

$$0 = t + \sum t_i z_i + \sum t_{i_1 i_2} z_{i_1} \cdot z_{i_2} + \sum t_{i_1 i_2 i_3} z_{i_1} \cdot z_{i_2} \cdot z_{i_3} + \\ + \cdots + t_{12 \ldots l} z_1 \cdot z_2 \cdot \cdots \cdot z_l. \tag{2.11$_0$}$$

Wir multiplizieren mit $z_1 \cdot z_2 \cdot \cdots \cdot z_l$; auf Grund des assoziativen Gesetzes und der Regeln (2.8), (2.9) verschwinden auf der rechten Seite alle Produkte, in denen ein Index zweimal auftritt, und es entsteht daher die Gleichung

$$0 = t \cdot z_1 \cdot z_2 \cdot \cdots \cdot z_l;$$

nach Satz Ia folgt hieraus—da der Koeffizientenbereich ein Körper ist—:

$$t = 0.$$

Wir betrachten einen Index i und multiplizieren (2.11$_0$) mit dem Produkt aller der z_j, für welche $j \neq i$ ist; aus analogen Gründen wie soeben entsteht:

$$0 = t_i \cdot z_1 \cdot z_2 \cdot \cdots \cdot z_l,$$

also folgt wie soeben:

$$t_i = 0.$$

Wir betrachten zwei Indizes i_1, i_2 und multiplizieren (2.11_0) mit dem Produkt aller der z_j, für welche $j \neq i_1$ und $j \neq i_2$ ist; es ergibt sich

$$t_{i_1 i_2} = 0.$$

So fortfahrend erkennt man, daß in der Tat alle Koeffizienten auf der rechten Seite von (2.11_0) gleich 0 sind.

Somit bilden die Elemente (2.10) eine Basis, und die Elemente von $\mathfrak{R}(\Gamma)$ lassen sich in eineindeutiger Weise mit den Ausdrücken (2.11) identifizieren. Daß für die Multiplikation die antikommutative Regel (2.8) gilt, wurde schon gezeigt. Für den vollständigen Beweis des Satzes I fehlt nur noch die Bestätigung der in Nr. 3 angegebenen Dimensions-Regeln.

Aus dem Satz Ia folgt, daß die Dimensionszahl $d(z_1 \cdot z_2 \cdot \cdots \cdot z_l) = d_0$ wohlbestimmt und $\geqq 0$ ist; für $i_1 < i_2 < \cdots < i_r$ ist $d(z_{i_1} \cdot z_{i_2} \cdot \cdots \cdot z_{i_r}) \geqq d_0$, also ist infolge von (2.11) auch $d(z) \geqq d_0$ für jedes Element z von $\mathfrak{R}(\Gamma)$; da es 0-dimensionale Elemente z gibt, ist daher $d_0 = 0$. Dies ist, wenn Γ die Dimension n hat, gleichbedeutend mit $\delta(z_1 \cdot z_2 \cdot \cdots \cdot z_l) = n$; wenn wir $\delta(z_i) = m_i$ setzen, ist daher nach (2.1)

$$n = m_1 + m_2 + \cdots + m_l .$$

Allgemein ergibt sich für $i_1 < i_2 < \cdots < i_r$, wieder nach (2.1),

$$\begin{aligned} d(z_{i_1} \cdot z_{i_2} \cdot \cdots \cdot z_{i_r}) &= n - \delta(z_{i_1} \cdot z_{i_2} \cdot \cdots \cdot z_{i_r}) \\ &= n - m_{i_1} - m_{i_2} - \cdots - m_{i_r} . \end{aligned}$$

Daß schließlich alle m_i ungerade sind, wurde schon durch (2.7) festgestellt.

Damit ist der Satz I vollständig bewiesen—unter der Voraussetzung, daß die Sätze Ia und Ib gelten, die im §4 bewiesen werden sollen.

16. *Hilfssätze.*[24] Wir stellen hier noch einige einfache Tatsachen zusammen, die wir später (§4) brauchen werden.

M sei eine n-dimensionale Mannigfaltigkeit und $(z_1, z_2, \cdots, z_l)$ ein beliebiges Erzeugenden-System von $\mathfrak{R}(M)$. Unter $\mathfrak{U}$ verstehen wir die Menge aller derjenigen Elemente, die sich in der Gestalt

$$w_2 \cdot z_2 + w_3 \cdot z_3 + \cdots + w_l \cdot z_l$$

schreiben lassen, wobei die w_i beliebige Elemente von $\mathfrak{R}(M)$ sind.

Wir behaupten:

(a) *Jedes homogen-dimensionale Element z mit $d(z_1) < d(z) < n$ gehört zu $\mathfrak{U}$.*

(b) *Ist z_1 in $\mathfrak{U}$ enthalten, so ist das Erzeugenden-System $(z_1, z_2, \cdots, z_l)$ reduzibel.*

Beweis von (a): Man schreibe z als lineare Verbindung von Potenzprodukten $z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdot \cdots \cdot z_l^{\alpha_l}$ (mit rationalen Koeffizienten); da z und alle z_i homogen-

[24] Die Hilfssätze der Nummern 16–18 und 20–22, die an und für sich kaum Interesse verdienen, werden erst in den Nummern 28 und 29 angewandt.

dimensional sind, kann man sich dabei offenbar auf solche Potenzprodukte beschränken, die selbst die Dimension $d(z)$ haben; für ein solches Potenzprodukt ist dann nach (2.1)

$$\alpha_1 \cdot \delta(z_1) + \alpha_2 \cdot \delta(z_2) + \cdots + \alpha_l \cdot \delta(z_l) = \delta(z);$$

da $\delta(z) < \delta(z_1)$ ist, folgt hieraus $\alpha_1 = 0$, und da $\delta(z) > 0$ ist, folgt weiter, daß *nicht* $\alpha_2 = \cdots = \alpha_l = 0$ ist; dies bedeutet: jedes Potenzprodukt enthält wenigstens eines der Elemente $z_2, \cdots, z_l$ als Faktor, d.h. z gehört zu $\mathfrak{U}$.

Beweis von (b): Es sei $z_1 \in \mathfrak{U}$, also

$$z_1 = \sum_{j=2}^{l} w_j \cdot z_j; \tag{2.12}$$

dabei kann man, da die z_i homogen-dimensional sind, auch die Elemente w_j als homogen-dimensional und $d(w_j \cdot z_j) = d(z_1)$, also $\delta(w_j) + \delta(z_j) = \delta(z_1)$, annehmen. Da die $\delta(z_j) > 0$ sind (Nr. 13), sind daher alle $\delta(w_j) < \delta(z_1)$. Hieraus folgt, analog wie im Beweis von (a): stellt man w_j als lineare Verbindung von Potenzprodukten $z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdot \cdots \cdot z_l^{\alpha_l}$ dar, so ist immer $\alpha_1 = 0$; jedes w_j ist also durch $z_2, \cdots, z_l$ allein auszudrücken, und nach (2.12) ist dann auch z_1 durch $z_2, \cdots, z_l$ auszudrücken. Aber dann erzeugen bereits $z_2, \cdots, z_l$ den Ring $\mathfrak{R}(M)$.

17. Es sei z ein homogen-dimensionales Element von $\mathfrak{R}(M)$ und $d(z) < n$. Unter $\mathfrak{V}$ verstehen wir die Menge aller derjenigen Elemente, die sich als Summen von Produkten $w \cdot v$ schreiben lassen, wobei die Elemente v homogen-dimensional mit

$$d(v) \neq d(z), \qquad d(v) < n \tag{2.13}$$

und die Elemente w beliebig sind.

Wir behaupten:

(c) *Ist z in $\mathfrak{V}$ enthalten, so ist z nicht maximal.*

Beweis: Es sei z in $\mathfrak{V}$ enthalten, also

$$z = \sum w_h \cdot v_h; \tag{2.14}$$

hierin sind die v_h homogen-dimensional und erfüllen (2.13); da z homogen-dimensional ist, dürfen wir auch die w_h als homogen-dimensional und $d(w_h \cdot v_h) = d(z)$, also $\delta(w_h) + \delta(v_h) = \delta(z)$, für alle h annehmen. Dann ist $\delta(v_h) \leqq \delta(z)$, also, da nach (2.13) $\delta(v_h) \neq \delta(z)$ ist, $\delta(v_h) < \delta(z)$, $d(v_h) > d(z)$. Da nach (2.13) andererseits $\delta(v_h) > 0$ ist, ist auch $\delta(w_h) < \delta(z)$, $d(w_h) > d(z)$. Aus $d(v_h) > d(z)$ und $d(w_h) > d(z)$ folgt nach (2.14), daß z nicht maximal ist.

18. *Bemerkung über Ideale in* $\mathfrak{R}(M)$. Da die Multiplikation in dem Ring $\mathfrak{R}(M)$ im allgemeinen nicht kommutativ ist, hat man unter den Idealen des Ringes zwischen Links-, Rechts- und zweiseitigen Idealen zu unterscheiden. Jedoch gilt folgender Satz:

Es seien $x_1, x_2, \cdots, x_m$ *homogen-dimensionale* Elemente von $\mathfrak{R}(M)$ und $\mathfrak{X}$ das von ihnen erzeugte Links-Ideal, d.h. die Menge aller Elemente der Gestalt

$$w_1 \cdot x_1 + w_2 \cdot x_2 + \cdots + w_m \cdot x_m \tag{2.15}$$

mit willkürlichen Elementen w_i; dann ist $\mathfrak{X}$ zugleich Rechts-Ideal, also *zweiseitig*.

Zum Beweis ziehen wir eine volle additive Basis $(Z_1, Z_2, \cdots, Z_q)$ von $\mathfrak{R}(M)$ heran, deren Elemente Z_h wir als homogen-dimensional annehmen dürfen. Dann ist die Menge $\mathfrak{X}$ aller Elemente (2.15) identisch mit der Menge aller linearen Verbindungen der Produkte $Z_h \cdot x_i$ mit rationalen Koeffizienten ($h = 1, 2, \cdots, q$; $i = 1, 2, \cdots, m$). Nach (2.2) ist aber, da die Z_h und die x_i homogen-dimensional sind, $Z_h \cdot x_i = \pm x_i \cdot Z_h$, und daher ist $\mathfrak{X}$ auch die Menge aller linearen Verbindungen der Produkte $x_i \cdot Z_h$, also die Menge aller Elemente

$$x_1 \cdot w_1 + x_2 \cdot w_2 + \cdots + x_m \cdot w_m$$

mit willkürlichen w_i; diese Menge ist aber das von $x_1, x_2, \cdots, x_m$ erzeugte Rechts-Ideal.

Auf Grund dieses Satzes sind insbesondere die Mengen $\mathfrak{U}$ und $\mathfrak{B}$, die wir in Nr. 16 und 17 betrachtet haben, *zweiseitige Ideale.*

§3. Eigenschaften des Ringes $\mathfrak{R}(M \times M)$.

In Nr. 8 wurde die Rolle angedeutet, welche die Produkt-Mannigfaltigkeit $\Gamma \times \Gamma$ beim Beweise des Satzes I spielt. Der gegenwärtige Paragraph enthält lediglich eine, auf diesen Zweck zugeschnittene, Zusammenstellung und Formulierung bekannter Eigenschaften des Ringes $\mathfrak{R}(M \times M)$; dabei bezeichnet M eine beliebige Mannigfaltigkeit. Der Koeffizientenbereich ist wie immer der Körper der rationalen Zahlen.[25]

19. Zu je zwei Elementen x, y von $\mathfrak{R}(M)$ gehört ein Element $x \times y$ von $\mathfrak{R}(M \times M)$. Diese Produktbildung ist mit der Addition distributiv verknüpft. Sind x und y homogen-dimensional, so ist auch $x \times y$ homogen-dimensional und $d(x \times y) = d(x) + d(y)$.

Umgekehrt läßt sich jedes Element von $\mathfrak{R}(M \times M)$ als $\sum (x_h \times y_h)$ darstellen; genauer: ist das System $(Z_1, Z_2, \cdots, Z_q)$ eine volle additive Basis von $\mathfrak{R}(M)$—d.h. die Vereinigung von Homologiebasen aller Dimensionen—, so bilden die $Z_i \times Z_k$ eine volle additive Basis von $\mathfrak{R}(M \times M)$; die Elemente von $\mathfrak{R}(M \times M)$ sind also in eineindeutiger Weise als $\sum t_{ik}(Z_i \times Z_k)$ mit rationalen t_{ik} auszudrücken. Damit sind die additiven Eigenschaften von $\mathfrak{R}(M \times M)$ gegeben.

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[25] Wegen der additiven Eigenschaften von $\mathfrak{R}(M \times M)$ vergl. man z.B. Alexandroff-Hopf, a.a.O.[6], Kap. VII, §3; jedoch hat man die dortigen Betrachtungen dadurch abzuändern (und wesentlich zu vereinfachen), daß man rationale Koeffizienten verwendet; wegen der multiplikativen Eigenschaften, insbesondere unserer Formel (3.1), vergl. man Lefschetz, a.a.O.[22], Chapter V, §3, insbesondere Formel (21).—Übrigens darf in unserem ganzen §3 der Koeffizientenbereich ein *beliebiger* Körper sein.

Die Multiplikation ist nun durch die Regel

$$(x \times y) \cdot (x' \times y') = (-1)^{\delta(x)\delta(y')}(x \cdot x' \times y \cdot y'), \tag{3.1}$$

welche *für homogen-dimensionale* x, y' gilt, und durch die distributiven Gesetze vollständig bestimmt.

20. *Hilfssätze.*[24] $\mathfrak{X}$ sei eine additive Untergruppe von $\mathfrak{R}(M)$, die rational abgeschlossen ist, d.h.: aus $x \in \mathfrak{X}$ folgt $tx \in \mathfrak{X}$ für jede rationale Zahl t. Ferner seien x, y, und zwar $y \neq 0$, zwei solche Elemente von $\mathfrak{R}(M)$, daß sich das Element $y \times x$ von $\mathfrak{R}(M \times M)$ in der Form

$$y \times x = \sum_h (y_h \times x_h) \tag{3.2}$$

darstellen läßt, wobei die x_h Elemente von $\mathfrak{X}$ sind, während von den Elementen y_h nichts vorausgesetzt wird.

Behauptung: *Dann ist* $x \in \mathfrak{X}$.

Beweis: $(Z_1, Z_2, \cdots, Z_q)$ sei eine volle additive Basis in $\mathfrak{R}(M)$. Es sei

$$x = \sum_i a_i Z_i, \qquad y = \sum_j b_j Z_j,$$

$$x_h = \sum_i c_{hi} Z_i, \qquad y_h = \sum_j d_{hj} Z_j$$

mit rationalen a, b, c, d; dann folgt aus (3.2), daß

$$b_j a_i = \sum_h d_{hj} c_{hi},$$

also

$$b_j x = \sum_h d_{hj} x_h \tag{3.3}$$

für jedes j ist. Da nach Voraussetzung $y \neq 0$ ist, ist wenigstens ein $b_j \neq 0$; es sei etwa $b_1 \neq 0$. Dann folgt aus (3.3)

$$x = \sum_h \frac{d_{h1}}{b_1} x_h \in \mathfrak{X}.$$

21. $\mathfrak{X}$ sei ein (zweiseitiges) Ideal in $\mathfrak{R}(M)$, welches von homogen-dimensionalen Elementen $x_1, x_2, \cdots, x_m$ erzeugt wird (man vergl. Nr. 18). Unter $\mathfrak{X}^*$ verstehen wir die Menge aller derjenigen Elemente von $\mathfrak{R}(M \times M)$, welche sich in der Gestalt

$$\sum_h (y_h \times x_h') \tag{3.4}$$

schreiben lassen, wobei die x_h' Elemente von $\mathfrak{X}$, die y_h beliebige Elemente von $\mathfrak{R}(M)$ sind.

Behauptung: $\mathfrak{X}^*$ *ist ein zweiseitiges Ideal in* $\mathfrak{R}(M \times M)$.

Beweis: Es sei wieder $(Z_1, Z_2, \cdots, Z_q)$ eine volle additive Basis in $\mathfrak{R}(M)$; dann ist $\mathfrak{X}$ identisch mit der Menge aller linearen Verbindungen der Produkte $Z_j \cdot x_i$ (mit rationalen Koeffizienten) und $\mathfrak{X}^*$ daher identisch mit der Menge aller linearen Verbindungen der Elemente $Z_k \times Z_j \cdot x_i$ $(j, k = 1, 2, \cdots, q;$ $i = 1, 2, \cdots, m)$. Da wir die Z_k als homogen-dimensional annehmen dürfen, ist aber nach (3.1)

$$Z_k \times Z_j x_i = \pm (Z_k \times Z_j) \cdot (1 \times x_i);$$

die $Z_k \times Z_j$ bilden eine volle additive Basis; folglich ist $\mathfrak{X}^*$ die Menge aller Summen

$$\sum_i W_i \cdot (1 \times x_i),$$

wobei W_i willkürliche Elemente von $\mathfrak{R}(M \times M)$ sind; dies bedeutet aber: $\mathfrak{X}^*$ ist das Links-Ideal, das von den Elementen $1 \times x_i$, $i = 1, 2, \cdots, m$, erzeugt wird. Da die x_i homogen-dimensional sind, sind auch die Elemente $1 \times x_i$ homogen-dimensional, und nach Nr. 18 ist $\mathfrak{X}^*$ daher ein zweiseitiges Ideal.

22. Wir bringen den hiermit bewiesenen Satz in Verbindung mit dem Satz aus Nr. 20. Ein Ideal $\mathfrak{X}$ hat die in Nr. 20 genannte Eigenschaft der rationalen Abgeschlossenheit, da man ja die rationalen Zahlen als die skalaren Elemente—d.h. die rationalen Vielfachen des Eins-Elementes—von $\mathfrak{R}(M)$ auffassen kann. Daher ergibt sich aus Nr. 20 und 21 das folgende *Lemma*:

Die Ideale $\mathfrak{X}$ und $\mathfrak{X}^*$ sollen wie in Nr. 21 *erklärt sein; ferner seien x und y Element von $\mathfrak{R}(M)$, für welche*

$$y \neq 0$$

$$y \times x \equiv 0 \qquad \text{mod } \mathfrak{X}^*$$

gilt; dann ist

$$x \equiv 0 \qquad \text{mod } \mathfrak{X}.$$

23. *Die Homomorphismen* Λ und P. M sei eine n-dimensionale Mannigfaltigkeit und $(Z_1, Z_2, \cdots, Z_q)$ eine volle additive Basis von $\mathfrak{R}(M)$; wir dürfen annehmen, daß alle Z_j homogen-dimensional sind, daß $Z_1 = 1$ und $d(Z_j) < n$ für $j > 1$ ist. Da die $Z_j \times Z_k$ eine Basis von $\mathfrak{R}(M \times M)$ bilden, besitzt jedes Element Q von $\mathfrak{R}(M \times M)$ eine und nur eine Darstellung

$$Q = \sum_{j,k} t_{jk}(Z_j \times Z_k) \tag{3.4}$$

mit rationalen t_{jk}. Wir setzen

$$\Lambda(Q) = \sum_k t_{1k} Z_k \tag{3.5}$$

Zum Beispiel ist für jedes Element y von $\mathfrak{R}(M)$

$$\Lambda(1 \times y) = y \tag{3.6}$$

und, falls $d(x) < n$ ist,

$$\Lambda(x \times y) = 0; \tag{3.7}$$

man bestätigt (3.6) und (3.7) einfach, indem man y, bzw. x und y, als lineare Verbindungen der Z_k schreibt.

Wir fassen Λ als Abbildung des Ringes $\mathfrak{R}(M \times M)$ in den Ring $\mathfrak{R}(M)$ auf und behaupten: Λ *ist eine Homomorphie.*

Daß Λ additiv homomorph, d.h. daß

$$\Lambda(Q_1 + Q_2) = \Lambda(Q_1) + \Lambda(Q_2)$$

ist, liest man unmittelbar aus (3.4) und (3.5) ab. Für den Beweis der multiplikativen Homomorphie, d.h. der Gültigkeit der Gleichheit

$$\Lambda(Q_1 \cdot Q_2) = \Lambda(Q_1) \cdot \Lambda(Q_2) \tag{3.8}$$

darf man sich infolge der Distributivität und der schon konstatierten additiven Homomorphie auf den Fall beschränken, daß Q_1, Q_2 Elemente der Basis $Z_j \times Z_k$ von $\mathfrak{R}(M \times M)$ sind. Es sei also

$$Q_1 = Z_h \times Z_i, \qquad Q_2 = Z_j \times Z_k.$$

Ist $h = j = 1$, so ist einerseits

$$\Lambda(Q_1) = Z_i, \qquad \Lambda(Q_2) = Z_k,$$

andererseits nach (3.1)

$$Q_1 \cdot Q_2 = 1 \times Z_i \cdot Z_k,$$

also nach (3.6)

$$\Lambda(Q_1 \cdot Q_2) = Z_i \cdot Z_k;$$

folglich gilt (3.8). Ist nicht $h = j = 1$, so ist wenigstens eines der Elemente $\Lambda(Q_1)$, $\Lambda(Q_2)$ gleich Null, also ist die rechte Seite von (3.8) Null; andererseits ist nach (3.1)

$$Q_1 \cdot Q_2 = \pm Z_h \cdot Z_j \times Z_i \cdot Z_k,$$

und hierin ist $d(Z_h \cdot Z_j) < n$; daher ist nach (3.7) auch die linke Seite $\Lambda(Q_1 \cdot Q_2)$ von (3.8) gleich Null.—Damit ist die Homomorphie-Eigenschaft von Λ bewiesen.

Setzt man im Anschluß an (3.4)

$$\mathsf{P}(Q) = \sum_j t_{j1} Z_j, \tag{3.9}$$

so ergibt sich ganz analog: P ist eine homomorphe Abbildung von $\mathfrak{R}(M \times M)$ in $\mathfrak{R}(M)$.

24. Aus (3.4), (3.5), (3.9) liest man ab, daß für jedes Element Q von $\mathfrak{R}(M \times M)$ Gleichungen

$$\begin{aligned} Q &= (1 \times \Lambda(Q)) + \sum_j (Z_j \times Y_j), \qquad d(Z_j) < n, \\ Q &= (\mathbf{P}(Q) \times 1) + \sum_k (X_k \times Z_k), \qquad d(Z_k) < n \end{aligned} \tag{3.10}$$

gelten.

Ferner ist klar: ist Q homogen $(n + r)$-dimensional, so sind $\Lambda(Q)$ und $\mathsf{P}(Q)$ homogen r-dimensional.

Es sei jetzt Q homogen $(n + r)$-dimensional und $r < n$. Dann ist in (3.4) $t_{11} = 0$, und daher erhält (3.4) mit Hilfe von (3.5) und (3.9) die Gestalt

$$Q = (1 \times \Lambda(Q)) + (\mathsf{P}(Q) \times 1) + \sum_{j=2}^{q} \sum_{k=2}^{q} t_{jk}(Z_j \times Z_k). \tag{3.11}$$

Da die Z_i homogen-dimensional sind, sind auch die $Z_j \times Z_k$ homogen-dimensional, und es ist $d(Z_j \times Z_k) = d(Z_j) + d(Z_k)$; daher sind in (3.11) nur solche $t_{jk} \neq 0$, für welche

$$d(Z_j) + d(Z_k) = n + r$$

ist; ferner ist in (3.11), da $j > 0$, $k > 0$ ist, immer

$$d(Z_j) < n, \qquad d(Z_k) < n$$

und folglich

$$d(Z_k) > r, \qquad d(Z_j) > r.$$

Mithin läßt sich (3.11) auch so ausdrücken:

$$\begin{cases} Q = (1 \times \Lambda(Q)) + (\mathsf{P}(Q) \times 1) + \sum_h (x_h \times y_h), \\ x_h, y_h \text{ homogen-dimensional mit} \\ r < d(x_h) < n, \qquad r < d(y_h) < n. \end{cases} \tag{3.12}$$

Wir fassen zusammen: *Es gibt zwei solche Homomorphismen Λ und P des Ringes $\mathfrak{R}(M \times M)$ in den Ring $\mathfrak{R}(M)$, daß jedes Element Q von $\mathfrak{R}(M \times M)$ Darstellungen* (3.10) *und daß jedes homogen $(n + r)$-dimensionale Element Q mit $r < n$ eine Darstellung* (3.12) *besitzt.*

Es ist übrigens leicht zu sehen, daß die Abbildungen Λ und P, die wir durch (3.5) und (3.9) unter Benutzung einer speziellen Basis $\{Z_i\}$ erklärt haben, von der Wahl dieser Basis unabhängig sind.

§4. Beweis des Satzes I.

25. *Der Umkehrungs-Homomorphismus.* M und M' seien beliebige Mannigfaltigkeiten,[2] und F sei eine Abbildung von M in M'. Dann bewirkt F eine Abbildung des Ringes $\mathfrak{R}(M)$ in den Ring $\mathfrak{R}(M')$; wir bezeichnen auch diese Ring-Abbildung mit F; sie ist übrigens additiv homomorph, jedoch im allgemeinen nicht multiplikativ homomorph.

Es gilt der Satz:[26]

[26] H. Hopf, *Zur Algebra der Abbildungen von Mannigfaltigkeiten* [Journ. f.d.r.u.a. Math. **163** (1930), 71–88].—*Neue Begründung und Verallgemeinerung des Umkehrungs-Homomorphismus*: H. Freudenthal, *Zum Hopfschen Umkehrhomomorphismus* [Annals of Math. (2) **38** (1937), 847–853]; ferner: A. Komatu, *Über die Ringdualität eines Kompaktums* [Tôhoku Math. Journ. **43** (1937), 414–420]; H. Whitney, *On products in a complex* [Annals of Math. (2) **39** (1938), 397–432], (Theorem 6).—Die Eigenschaft 3 unseres Textes ist in meiner zitierten Arbeit nicht hervorgehoben, da dort nur gleichdimensionale Mannigfaltigkeiten betrachtet werden; sie ergibt sich aber unmittelbar aus jeder einzelnen der verschiedenen, in den soeben genannten Arbeiten enthaltenen, Definitionen von Φ; überdies ist sie eine Folge der Eigenschaft 2; hierzu vergl. man Nr. 11 meiner Arbeit *"Ein topologischer Beitrag zur reellen Algebra"* [Com. Math. Helvet.; erscheint nächstens].

Es existiert eine Abbildung Φ des Ringes $\mathfrak{R}(M')$ in den Ring $\mathfrak{R}(M)$ mit den folgenden drei Eigenschaften:

1) *Φ ist ein additiver und multiplikativer Homomorphismus;*

2) *Φ ist mit F durch die Funktionalgleichung*

$$F(\Phi(z)\cdot x) = z\cdot F(x) \tag{4.1}$$

verknüpft, in welcher x ein beliebiges Element von $\mathfrak{R}(M)$ und z ein beliebiges Element von $\mathfrak{R}(M')$ ist;

3) *ist z homogen-dimensional, so ist auch $\Phi(z)$ homogen-dimensional, und zwar ist $\delta(\Phi(z)) = \delta(z)$,* also

$$d(\Phi(z)) = d(z) + d(M) - d(M').$$

Φ heißt der "Umkehrungs-Homomorphismus" von F.

26. *Ansatz zum Beweis der Sätze* Ia *und* Ib. In der n-dimensionalen Mannigfaltigkeit Γ sei eine stetige Multiplikation gegeben (Nr. 1). Da wir die Punktepaare (p, q) von Γ als die Punkte $p \times q$ der Produkt-Mannigfaltigkeit $\Gamma \times \Gamma$ deuten können, ist die stetige Multiplikation gleichbedeutend mit einer Abbildung F von $\Gamma \times \Gamma$ in Γ; diese ist durch $F(p \times q) = pq$ bestimmt.

Wie in Nr. 25 bezeichnen wir auch die durch F bewirkte Abbildung des Ringes $\mathfrak{R}(\Gamma \times \Gamma)$ in den Ring $\mathfrak{R}(\Gamma)$ mit F. Das Element von $\mathfrak{R}(\Gamma)$, das durch einen einfach gezählten Punkt repräsentiert wird, heiße p. Das mit einer rationalen Zahl c multiplizierte Eins-Element von $\mathfrak{R}(\Gamma)$ bezeichnen wir kurz mit c. Dann sind die Grade c_l und c_r, die in Nr. 1 erklärt worden sind, offenbar durch

$$F(p \times 1) = c_l, \qquad F(1 \times p) = c_r \tag{4.2}$$

charakterisiert.

Φ sei der Umkehrungs-Homomorphismus von F. Dann folgt aus (4.1) und (4.2)

$$F(\Phi(z)\cdot(p \times 1)) = c_l z, \tag{4.3a}$$

$$F(\Phi(z)\cdot(1 \times p)) = c_r z \tag{4.3b}$$

für jedes Element z von $\mathfrak{R}(\Gamma)$.

Die Homomorphismen Λ und P sind in Nr. 23, 24 erklärt worden; wir setzen

$$\Lambda\Phi(z) = \lambda(z), \qquad \mathsf{P}\Phi(z) = \rho(z)$$

für jedes Element z von $\mathfrak{R}(\Gamma)$; dann sind λ und ρ *Homomorphismen des Ringes* $\mathfrak{R}(\Gamma)$ *in sich,* und nach Nr. 24, (3.10), gelten für jedes Element z von $\mathfrak{R}(\Gamma)$ Gleichungen

$$\Phi(z) = (1 \times \lambda(z)) + \sum_i (Z_i \times Y_i), \qquad d(Z_i) < n, \tag{4.4a}$$

$$\Phi(z) = (\rho(z) \times 1) + \sum_k (X_k \times Z_k), \qquad d(Z_k) < n. \tag{4.4b}$$

Nach Nr. 25, 3), ordnet Φ, da $d(\Gamma \times \Gamma) = 2n$, $d(\Gamma) = n$ ist, jedem homogen r-dimensionalen Element von $\mathfrak{R}(\Gamma)$ ein homogen $(n + r)$-dimensionales Element von $\mathfrak{R}(\Gamma \times \Gamma)$ zu; nach Nr. 24 ordnen Λ und P den homogen $(n + r)$-dimensionalen Elementen von $\mathfrak{R}(\Gamma \times \Gamma)$ homogen r-dimensionale Elemente von $\mathfrak{R}(\Gamma)$ zu; folglich sind λ und ρ *dimensionstreu.*

27. Jetzt sei

$$c_l \neq 0, \qquad c_r \neq 0, \tag{4.5}$$

also Γ eine Γ-Mannigfaltigkeit. Wir behaupten: *dann sind* λ *und* ρ *Automorphismen*[27] *von* $\mathfrak{R}(\Gamma)$.

Beweis: Ist $\lambda(z) = 0$, so ist nach (4.4a)

$$\Phi(z) = \sum (Z_i \times Y_i) \text{ mit } d(Z_i) < n;$$

aus $d(Z_i) < n$ folgt $Z_i \cdot p = 0$; folglich ist, nach (3.1),

$$\Phi(z) \cdot (p \times 1) = 0,$$

also nach (4.3a)

$$c_l z = 0,$$

also nach (4.5), da der Koeffizientenbereich ein Körper ist,

$$z = 0.$$

Das bedeutet: der Homomorphismus λ ist eineindeutig. Folglich ist die Determinante der Substitution, welche durch λ auf eine volle additive Basis von $\mathfrak{R}(\Gamma)$ ausgeübt wird, nicht Null. Da der Koeffizientenbereich ein Körper ist, folgt hieraus: λ ist eine Abbildung von $\mathfrak{R}(\Gamma)$ *auf* sich. Somit ist λ ein Automorphismus.—Analog beweist man die Behauptung für ρ.

Wir fassen zusammen, indem wir noch das Ergebnis von Nr. 24 heranziehen: *Für jede n-dimensionale* Γ-*Mannigfaltigkeit* Γ *sind zwei dimensionstreue Automorphismen* λ *und* ρ *von* $\mathfrak{R}(\Gamma)$ *und ein Homomorphismus* Φ *von* $\mathfrak{R}(\Gamma)$ *in* $\mathfrak{R}(\Gamma \times \Gamma)$ *ausgezeichnet; ist z ein homogen r-dimensionales Element von* $\mathfrak{R}(\Gamma)$ *und* $r < n$, *so gilt*

$$\left\{\begin{array}{l} \Phi(z) = (1 \times \lambda(z)) + (\rho(z) \times 1) + \sum (x_h \times y_h), \\ x_h, y_h \textit{ homogen-dimensional mit} \\ \qquad r < d(x_h) < n, \qquad r < d(y_h) < n. \end{array}\right. \tag{4.6}$$

In diesem Satz sind alle Eigenschaften der Γ-Mannigfaltigkeiten enthalten, die wir im Folgenden benutzen werden.

28. *Beweis des Satzes* Ia (Nr. 15). Es sei $(z_1, z_2, \cdots, z_l)$ ein irreduzibles Erzeugenden-System für die n-dimensionale Γ-Mannigfaltigkeit Γ. Wir werden

[27] Unter einem "Automorphismus" eines Ringes $\mathfrak{R}$ ist ein Auto-*Iso*morphismus von $\mathfrak{R}$ *auf* sich zu verstehen.

durch vollständige Induktion nach k beweisen: das Produkt von je k voneinander verschiedenen Elementen dieses Systems ist $\neq 0$. Für $k = 1$ ist dies richtig, da ein irreduzibles Erzeugenden-System niemals die Null enthalten kann. Es sei für $k - 1$ bewiesen, und k Elemente des Systems, etwa $z_1, z_2, \cdots, z_k$, seien vorgelegt; da ihre Anordnung lediglich das Vorzeichen ihres Produktes beinflussen kann, dürfen wir annehmen, daß

$$d(z_1) \leqq d(z_j), \qquad j = 1, 2, \cdots, k \tag{4.7}$$

ist; nach Induktions-Annahme ist

$$z_2 \cdot z_3 \cdot \cdots \cdot z_k \neq 0,$$

also, da ρ ein Automorphismus ist, auch

$$\rho(z_2 \cdot z_3 \cdot \cdots \cdot z_k) \neq 0. \tag{4.8}$$

Wir setzen $\lambda(z_i) = z_i'$ für $i = 1, 2, \cdots, l$. Da λ ein Automorphismus ist, bilden auch $z_1', z_2', \cdots, z_l'$ ein irreduzibles Erzeugenden-System. Da λ dimensionstreu ist, ist

$$d(z_i') = d(z_i), \qquad i = 1, 2, \cdots, l \tag{4.9}$$

und nach (4.7)

$$d(z_1') \leqq d(z_j'), \qquad j = 1, 2, \cdots, k. \tag{4.7'}$$

Wir verstehen nun wie in Nr. 16—mit dem Unterschied, daß wir die dortigen z_i durch unsere neuen z_i' ersetzen—unter $\mathfrak{U}$ das Ideal in $\mathfrak{R}(\Gamma)$, das von $z_2', \cdots, z_l'$ erzeugt wird (man vergl. Nr. 18); ferner verstehen wir unter $\mathfrak{U}^*$ das, analog wie in Nr. 21 erklärte, zu $\mathfrak{U}$ gehörige Ideal in $\mathfrak{R}(\Gamma \times \Gamma)$, also die Menge derjenigen Elemente von $\mathfrak{R}(\Gamma \times \Gamma)$, die sich in der Gestalt $\sum (w_h \times u_h')$ mit $u_h' \in \mathfrak{U}$ schreiben lassen.

Schreiben wir für $j = 1, 2, \cdots, k$ das Element $\Phi(z_j)$ in der Form (4.6):

$$\Phi(z_j) = (1 \times z_j') + (\rho(z_j) \times 1) + \sum (x_h \times y_h), \tag{4.6$_j$}$$

so lautet auf Grund von (4.9) und (4.7′) die eine der Dimensions-Bedingungen aus (4.6)

$$d(z_1') \leqq d(z_j') < d(y_h) < n;$$

folglich gehören nach Nr. 16 (a) alle in den Gleichungen (4.6$_j$) auftretenden y_h zu $\mathfrak{U}$ und daher die Summe $\sum (x_h \times y_h)$ zu $\mathfrak{U}^*$; für $1 < j \leqq k$ gehört außerdem z_j' zu $\mathfrak{U}$, also $1 \times z_j'$ zu $\mathfrak{U}^*$. Daher sind in den Gleichungen (4.6$_j$) die folgenden Kongruenzen modulo $\mathfrak{U}^*$ enthalten:

$$\Phi(z_1) \equiv (1 \times z_1') + (\rho(z_1) \times 1),$$

$$\Phi(z_j) \equiv \qquad (\rho(z_j) \times 1), \qquad j = 2, \cdots, k.$$

Da $\mathfrak{U}^*$ ein zweiseitiges Ideal ist, dürfen wir diese Kongruenzen miteinander multiplizieren, und da Φ und ρ Homomorphismen sind, ergibt sich dabei, bei Beachtung von (3.1):

$$\Phi(z_1 \cdot z_2 \cdot \cdots \cdot z_k) \equiv (\rho(z_2 \cdot \cdots \cdot z_k) \times z_1') + (\rho(z_1 \cdot z_2 \cdot \cdots \cdot z_k) \times 1).$$

Wäre nun $z_1 \cdot z_2 \cdot \cdots \cdot z_k = 0$, so würde hieraus

$$\rho(z_2 \cdot \cdots \cdot z_k) \times z_1' \equiv 0 \bmod \mathfrak{U}^*$$

folgen; nach (4.8) und dem Lemma von Nr. 22 wäre dann

$$z_1' \equiv 0 \bmod \mathfrak{U},$$

also z_1' in $\mathfrak{U}$ enthalten; nach Nr. 16 (b) ist dies mit der Irreduzibilität des Erzeugenden-Systems $(z_1', z_2', \cdots, z_l')$ nicht verträglich. Folglich ist

$$z_1 \cdot z_2 \cdot \cdots \cdot z_k \neq 0,$$

was zu beweisen war.[28]

29. *Beweis des Satzes* Ib (Nr. 15). Im Ring der n-dimensionalen Γ-Mannigfaltigkeit Γ sei z ein homogen-dimensionales Element, für welches $d(z) < n$ und $\delta(z)$ *gerade* ist; wir haben zu zeigen, daß dann z nicht maximal ist.

Das Ideal $\mathfrak{V}$ sei wörtlich wie in Nr. 17 definiert; $\mathfrak{V}^*$ sei das Ideal in $\mathfrak{R}(\Gamma \times \Gamma)$, das analog wie in Nr. 21 durch $\mathfrak{V}$ bestimmt ist: es besteht aus allen Elementen $\sum (w_h \times v_h)$ mit $v_h \in \mathfrak{V}$.

In der Gleichung (4.6) für unser Element z gehören alle Elemente y_h zu $\mathfrak{V}$ und somit gehört die Summe $\sum (x_h \times y_h)$ zu $\mathfrak{V}^*$. Es gilt also die Kongruenz

$$\Phi(z) \equiv (1 \times \lambda(z)) + (\rho(z) \times 1) \quad \bmod \mathfrak{V}^*. \tag{4.10}$$

Nach (3.1) ist

$$(1 \times \lambda(z)) \cdot (\rho(z) \times 1) = \rho(z) \times \lambda(z) \tag{4.11}$$

und, da $\delta(z)$ gerade ist, auch

$$(\rho(z) \times 1) \cdot (1 \times \lambda(z)) = \rho(z) \times \lambda(z). \tag{4.11'}$$

Aus (4.11) und (4.11') ergibt sich, daß man die rechte Seite von (4.10) nach der binomischen Formel potenzieren kann; tut man dies und beachtet man, daß Φ, ρ, λ Homomorphismen sind, so erhält man für jeden positiven Exponenten m:

$$\Phi(z^m) \equiv \sum_{r=0}^{m} \binom{m}{r} (\rho(z^{m-r}) \times \lambda(z^r)) \qquad \bmod \mathfrak{V}^*. \tag{4.12}$$

Nun ist, da $d(z) < n$ ist, für $r > 1$ immer $d(z^r) < d(z)$, also, da λ dimensionstreu ist, auch $d(\lambda(z^r)) < d(z)$ und daher $\lambda(z^r) \in \mathfrak{V}$ und

$$\rho(z^{m-r}) \times \lambda(z^r) \in \mathfrak{V}^* \qquad \text{für } r > 1;$$

[28] Der Beweis zeigt, daß für den Satz Ia der Koeffizientenbereich ein *beliebiger Körper* sein darf.

daher reduziert sich die Kongruenz (4.12) auf

$$\Phi(z^m) \equiv m(\rho(z^{m-1}) \times \lambda(z)) + (\rho(z^m) \times 1) \bmod \mathfrak{B}^*. \tag{4.13}$$

Dies gilt für jeden positiven Exponenten m. Da aus Dimensions-Gründen $z^m = 0$ für hinreichend große m ist, kann man m so wählen, daß

$$z^{m-1} \neq 0, \qquad z^m = 0$$

ist. Dann entsteht aus (4.13) die Kongruenz

$$m(\rho(z^{m-1}) \times \lambda(z)) \equiv 0 \bmod \mathfrak{B}^*,$$

also, da der Koeffizientenbereich der Körper der rationalen Zahlen ist,[29]

$$\rho(z^{m-1}) \times \lambda(z) \equiv 0 \bmod \mathfrak{B}^*. \tag{4.14}$$

Da $z^{m-1} \neq 0$ und ρ ein Automorphismus ist, ist auch $\rho(z^{m-1}) \neq 0$; nach dem Lemma in Nr. 22 folgt daher aus (4.14)

$$\lambda(z) \equiv 0 \bmod \mathfrak{B},$$

d.h. $\lambda(z) \in \mathfrak{B}$. Der Automorphismus λ ist dimensionstreu, und die Menge $\mathfrak{B}$ ist durch Dimensions-Eigenschaften definiert; daher geht $\mathfrak{B}$ bei Ausübung von λ^{-1} in sich über, und mit $\lambda(z)$ ist daher auch z in $\mathfrak{B}$ enthalten. Nach dem Hilfssatz Nr. 17 (c) ist mithin z nicht maximal—was zu beweisen war.

Mit den Sätzen Ia und Ib ist der Satz I vollständig bewiesen (Nr. 15).

§5. Maximale und minimale Elemente. Beweis des Satzes III

30. *Die Ränge l_s.* Wir knüpfen an Nr. 13 und 14 an. Es sei wieder M eine beliebige n-dimensionale Mannigfaltigkeit. Die additive Gruppe der homogen r-dimensionalen Elemente von $\mathfrak{R}(M)$, also die r-te Bettische Gruppe (in bezug auf rationale Koeffizienten) heiße $\mathfrak{B}_r$; ihr Rang p_r ist die r-te Bettische Zahl von M.

Für $0 \leqq r < n$ verstehen wir unter $\mathfrak{U}_r$ die Gruppe derjenigen Elemente von $\mathfrak{B}_r$, welche sich aus den Elementen der Gruppe

$$\mathfrak{B}_n + \mathfrak{B}_{n-1} + \cdots + \mathfrak{B}_{r+1}$$

durch Multiplikation und Addition erzeugen lassen, mit anderen Worten: welche nicht maximal sind. $\mathfrak{U}_r$ läßt sich auch als die Gesamtheit derjenigen Elemente charakterisieren, die sich in der Form $\sum_i x_i \cdot y_i$ schreiben lassen, wobei x_i, y_i homogen-dimensional mit

$$\delta(x_i) + \delta(y_i) = n - r, \qquad 0 < \delta(x_i), \qquad 0 < \delta(y_i)$$

sind.

Den Rang von $\mathfrak{U}_r$ nennen wir q_r, und wir setzen

$$p_{n-s} - q_{n-s} = l_s, \qquad s = 1, 2, \cdots, n.$$

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[29] Dies ist die einzige Stelle in der ganzen Arbeit, an welcher benutzt wird, daß der Koeffizientenkörper die *Charakteristik* 0 hat.

Die Zahl l_s ist der Rang der Restklassengruppe $\mathfrak{B}_{n-s} - \mathfrak{U}_{n-s}$ und hat daher die folgende Bedeutung: es gibt ein System von l_s derartigen maximalen Elementen $y_1, y_2, \cdots, y_{l_s}$ mit $\delta(y_i) = s$, daß auch jede lineare Verbindung

$$u_1y_1 + u_2y_2 + \cdots + u_{l_s}y_{l_s}$$

mit rationalen Koeffizienten u_i, abgesehen von derjenigen mit $u_1 = \cdots = u_{l_s} = 0$, selbst maximal ist; dagegen gibt es für kein $l' > l_s$ ein System von l' derartigen Elementen.

31. Wir behaupten erstens: In einem Erzeugenden-System $(z_1, \cdots, z_L)$ von $\mathfrak{R}(M)$ ist die Anzahl der z_j mit $\delta(z_j) = s$ stets $\geqq l_s$.

Beweis: Es seien $y_1, \cdots, y_{l_s}$ maximale Elemente mit der soeben genannten Eigenschaft. Stellt man die y_i als Polynome in den Erzeugenden $z_1, \cdots, z_L$ dar, so haben diese Darstellungen aus Dimensions-Gründen die Gestalt

$$y_i = t_{i1}z_1 + \cdots + t_{im}z_m + Y_i\,;$$

dabei sind $z_1, \cdots, z_m$ diejenigen Erzeugenden z_j, für welche $\delta(z_j) = s$ ist $(m \geqq 0)$, t_{ij} rationale Zahlen und Y_i Polynome in den z_k mit $d(z_k) > n - s$. Wäre $m < l_s$, so besäße das Gleichungssystem

$$\sum_{i=1}^{l_s} u_i t_{ij} = 0, \qquad j = 1, \cdots, m,$$

eine rationale Lösung $(u_1, \cdots, u_{l_s}) \neq (0, \cdots, 0)$, und es wäre

$$u_1y_1 + \cdots + u_{l_s}y_{l_s} = u_1Y_1 + \cdots + u_{l_s}Y_{l_s}\,;$$

dieses Element wäre, wie die rechte Seite zeigt, nicht maximal—entgegen der Voraussetzung über die y_i.

Wir behaupten zweitens: Ist—bei Benutzung derselben Bezeichnungen wie soeben—$m > l_s$, so ist das Erzeugenden-System $(z_1, \cdots, z_L)$ reduzibel.

Beweis: Da $l' = m > l_s$ ist, kann man, wie am Schluß von Nr. 30 festgestellt wurde, Zahlen $u_1, \cdots, u_m$ so bestimmen, daß das Element

$$Z = u_1z_1 + \cdots + u_mz_m$$

nicht maximal ist, und daß nicht alle $u_j = 0$ sind; dann ist Z ein Polynom in den z_k mit $d(z_k) > n - s$, also erst recht in den z_i mit $i > m$, und man kann, wenn etwa $u_1 \neq 0$ ist, z_1 durch die z_i mit $i > 1$ ausdrücken und somit das Erzeugenden-System reduzieren.

Damit haben wir festgestellt: *In jedem irreduziblen Erzeugenden-System* $(z_1, \cdots, z_l)$ *ist die Anzahl derjenigen* z_i, *für welche* $\delta(z_i) = s$ *ist, gleich* l_s; *die Anzahl aller Erzeugenden eines irreduziblen Systems ist daher immer*

$$l = l_1 + \cdots + l_n.$$ [30]

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[30] Hierdurch ist jeder Mannigfaltigkeit M eine Invariante l zugeordnet, welche etwa der "Rang" von M heißen möge. Für eine Γ-Mannigfaltigkeit ist, wie sich aus dem Satz (b) in Nr. 4 und seiner Herleitung ergibt, die Summe der Bettischen Zahlen gleich 2^l; andererseits ist für eine Gruppen-Mannigfaltigkeit, wie Cartan mit Hilfe der Integral-Invarianten berechnet hat[3], (b), 24, der hier auftretende Exponent gleich dem "Rang" der *Gruppe*,

32. *Die minimalen Elemente.* Es sei v ein homogen-dimensionales Element von $\mathfrak{R}(M)$ und $d(v) > 0$. Wir betrachten seine Vielfachen, d.h. die Elemente $x \cdot v$, wobei x die Elemente von $\mathfrak{R}(M)$ durchläuft; unter diesen Vielfachen sind außer dem Null-Element erstens alle rationalen Vielfachen von v enthalten—das sind Elemente der Dimension $d(v)$—und zweitens, falls nur $v \neq 0$ ist, nach dem Poincaré-Veblenschen Dualitäts-Satz alle Elemente der Dimension 0; für alle Vielfachen ist $d(x \cdot v) \leqq d(v)$. Wir definieren nun: das Element v heißt "minimal", wenn es keine Vielfachen $x \cdot v$ besitzt, für welche

$$x \cdot v \neq 0, \qquad 0 < d(x \cdot v) < d(v)$$

ist. Diese Definition ist offenbar gleichwertig mit der folgenden: v ist minimal, wenn für alle homogen-dimensionalen x, für welche

$$0 < \delta(x) < d(v), \text{ also } n - d(v) < d(x) < n,$$

ist, die Produkte $x \cdot v = 0$ sind.

Dabei ist, wie oben gesagt, $d(v) > 0$ vorausgesetzt; diese Verabredung ist analog der früher getroffenen, die n-dimensionalen Elemente nicht als maximal zu bezeichnen. Dagegen ist das Element 0 des Ringes $\mathfrak{R}(M)$ minimal.[31]

33. *Ein Dualitätssatz.* Die Gruppen $\mathfrak{U}_r$ sind in Nr. 30 erklärt worden. Wir behaupten:

Das homogen s-dimensionale Element v ist dann und nur dann minimal, wenn es ein Annullator der Gruppe $\mathfrak{U}_{n-s}$ ist, d.h. wenn $u \cdot v = 0$ für jedes Element u aus $\mathfrak{U}_{n-s}$ gilt ($s = 1, \cdots, n$).

Beweis: Es sei erstens v minimal und $u \in \mathfrak{U}_{n-s}$; dann ist $u = \sum x_i \cdot y_i$, wobei die x_i, y_i homogen-dimensional sind, mit

$$\delta(x_i) + \delta(y_i) = s, \qquad \delta(x_i) > 0, \qquad \delta(y_i) > 0,$$

also

$$0 < \delta(y_i) < s = d(v);$$

hieraus folgt $y_i \cdot v = 0$ für jedes i, also $u \cdot v = 0$. Es sei zweitens v homogen s-dimensional, aber nicht minimal; dann gibt es ein homogen-dimensionales y mit $0 < \delta(y) < s$ und $y \cdot v \neq 0$, und nach dem Poincaré-Veblenschen Dualitätssatz gibt es dann weiter ein homogen-dimensionales x mit $\delta(x) = d(y \cdot v)$ und $x \cdot y \cdot v \neq 0$; da $\delta(x) = n - \delta(y) - \delta(v)$, also $\delta(xy) = n - s$ ist, ist $x \cdot y \in \mathfrak{U}_{n-s}$; somit ist v nicht Annullator von $\mathfrak{U}_{n-s}$.

Aus der damit bewiesenen Charakterisierung der minimalen Elemente als

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d.h. gleich der Dimension der maximalen Abelschen Untergruppen. Die Aufgabe, die Gleichheit zwischen dem Rang einer Gruppen-Mannigfaltigkeit—in dem Sinne, wie wir l oben für jede Mannigfaltigkeit M erklärt haben—und der genannten Dimensionszahl auf möglichst rein geometrischem Wege aufzuklären, habe ich in einer Arbeit behandelt, die in der Com. Math. Helvet. 1941 erscheint.

[31] Alle homogen 1-dimensionalen Elemente sind offenbar minimal.

Annullatoren ist erstens ersichtlich, daß sie eine additive *Gruppe* $\mathfrak{B}_s$ bilden—(dies kann man auch direkt im Anschluß an die Definition in Nr. 32 leicht feststellen)—, und zweitens erkennt man mit Hilfe des Poincaré-Veblenschen Dualitätssatzes jetzt ohne Mühe, daß der Rang dieser Gruppe $\mathfrak{B}_s$ gleich $p_{n-s} - q_{n-s}$ (Nr. 30), also gleich l_s ist. Damit ist der folgende Satz bewiesen, der eine neue Charakterisierung der Zahlen l_s enthält:

Die s-dimensionalen minimalen Elemente von $\mathfrak{R}(M)$ bilden eine additive Gruppe vom Range l_s.

34. *Ein Invarianzsatz.* Die Eigenschaft der "Minimalität" ist invariant gegenüber beliebigen stetigen Abbildungen; genauer: M und M' seien beliebige Mannigfaltigkeiten, und F sei eine Abbildung von M in M'; die dadurch bewirkte Abbildung von $\mathfrak{R}(M)$ in $\mathfrak{R}(M')$ nennen wir ebenfalls F. *Dann ist für jedes minimale Element v von $\mathfrak{R}(M)$ das Bild $F(v)$ minimales Element von $\mathfrak{R}(M')$.*

Beweis: x sei ein homogen-dimensionales Element von $\mathfrak{R}(M)$, und das Element $F(x)$ von $\mathfrak{R}(M')$ sei nicht minimal; wir haben zu zeigen, daß x nicht minimal ist. Falls $d(x) = 0$ ist, ist nichts zu beweisen; es sei $d(x) > 0$; dann ist, da die Abbildung F von $\mathfrak{R}(M)$ in $\mathfrak{R}(M')$ dimensionstreu ist, auch $d(F(x)) > 0$; da $F(x)$ nicht minimal ist, gibt es ein solches homogen-dimensionales Element z von $\mathfrak{R}(M')$, daß $0 < \delta(z) < d(F(x)) = d(x)$ und $z \cdot F(x) \neq 0$ ist. Bezeichnet Φ den Umkehrungs-Homomorphismus von F (Nr. 25), so folgt aus (4.1), daß auch $\Phi(z) \cdot x \neq 0$ ist; dies bedeutet, da nach Nr. 25, 3), $\delta(\Phi(z)) = \delta(z)$, also $0 < \delta(\Phi(z)) < d(x)$ ist: x ist nicht minimal.

35. *Sphärenbilder.* Für die s-dimensionale Sphäre S_s ist dasjenige Element von $\mathfrak{R}(S_s)$, das durch den Grundzyklus repräsentiert wird—also das Eins-Element—minimal; daher ist in dem soeben bewiesenen Satz der folgende enthalten—(statt M' schreiben wir M)—: *Für eine beliebige Mannigfaltigkeit M ist ein Element von $\mathfrak{R}(M)$, das durch das stetige Bild einer Sphäre in M repräsentiert wird, stets ein minimales Element.*[32]

Hieraus folgt weiter, da nach Nr. 33 die Gruppe $\mathfrak{B}_s$ den Rang l_s hat: *Ist $l_s = 0$, so ist in M jedes stetige Bild der s-dimensionalen Sphäre homolog* 0.[32]

36. *Anwendung auf Γ-Mannigfaltigkeiten.* Nach Satz Ib gibt es in einer Γ-Mannigfaltigkeit kein maximales Element z mit geradem $\delta(z)$; daher sind die Ränge l_s für alle geraden s gleich 0; aus Nr. 35 folgt mithin der *Satz III* (Nr. 9).[32]

Für ein Sphärenprodukt

$$S_{m_1} \times S_{m_2} \times \cdots \times S_{m_l}, \qquad d(S_{m_i}) = m_i,$$

gibt die Zahl l_s an, wieviele der Zahlen m_i gleich s sind; dies folgt aus der Charakterisierung der l_s am Schluß von Nr. 31 und der Tatsache, die man leicht

[32] An die Stelle einer wirklichen Sphäre darf offenbar auch eine Homologie-Sphäre (in bezug auf den rationalen Koeffizientenbereich) treten, d.h. eine Mannigfaltigkeit, welche dieselben Bettischen Zahlen hat wie eine Sphäre.

bestätigt, daß die Elemente $Z_1, \cdots, Z_l$, die in Nr. 3, Formel (*), angegeben sind, ein irreduzibles Erzeugenden-System bilden, und daß $\delta(Z_i) = m_i$ ist. Für eine Γ-Mannigfaltigkeit Γ gibt also l_s an, wieviele Faktor-Sphären in dem Sphärenprodukt, das einen mit Γ isomorphen Ring hat, s-dimensional sind. Hierin ist enthalten, daß die Exponenten l_s in Nr. 4, Formel (1′), mit unseren Rängen l_s übereinstimmen.

37. *Die Pontrjaginsche Multiplikation und eine Vermutung.* Die Grundlage für Pontrjagins Untersuchung der Homologie-Eigenschaften geschlossener Liescher Gruppen[15] ist die folgende "Produktbildung": In der Mannigfaltigkeit M sei eine stetige Multiplikation erklärt; x und y seien zwei Zyklen in M; werden x und y von zwei Punkten p bzw. q durchlaufen, so durchläuft das stetige Produkt pq einen Zyklus, den wir mit $x \bigcirc y$ bezeichnen wollen; variieren x und y in ihren Homologieklassen, so ändert sich die Homologieklasse von $x \bigcirc y$ nicht; wir dürfen daher $x \bigcirc y$ als Produkt zweier Homologieklassen x und y deuten; in unserer Ausdrucksweise aus Nr. 26 ist

$$x \bigcirc y = F(x \times y).$$

Diese Produktbildung ist mit der Addition der Homologieklassen offenbar distributiv verknüpft; nehmen wir nun weiter an, die stetige Multiplikation in M sei *assoziativ*, dann wird auch die Produktbildung $x \bigcirc y$ assoziativ sein. Somit sieht man: Für eine *Gruppen*-Mannigfaltigkeit Γ läßt sich die additive Gruppe der Homologieklassen nicht nur durch die Schnittbildung $x \cdot y$ zu dem Ring $\mathfrak{R}(\Gamma)$, sondern außerdem durch die Pontrjaginsche Produktbildung $x \bigcirc y$ zu einem Ring $\mathfrak{P}(\Gamma)$ machen.

Die Bedeutung des Ringes $\mathfrak{P}(\Gamma)$ liegt auf der Hand: in seiner Struktur äußert sich die Wirkung der Gruppen-Multiplikation auf die Zyklen—also ein Zusammenhang zwischen den algebraischen Eigenschaften der Gruppe einerseits, den geometrischen Eigenschaften der Mannigfaltigkeit andererseits. Die Untersuchung dieses Ringes ist daher gewiß eine wichtige Aufgabe. Ich will hierzu eine Vermutung äußern, die sowohl durch Pontrjagins Ergebnisse, als auch durch unseren Satz I nahegelegt wird.[33]

Auch in jedem Sphären-Produkt

$$H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_l}$$

läßt sich neben der Schnittbildung in naheliegender Weise noch eine zweite Multiplikation, die "Aufspannung", erklären. Wie in Nr. 3 bezeichnen wir den Grundzyklus der Sphäre S_m selbst mit S_m und mit p immer einen einfach gezählten Punkt; wir stellen den Elementen (*) aus Nr. 3 die folgenden Elemente gegenüber:

$$(**) \qquad \begin{aligned} V_1 &= S_{m_1} \times p \times p \times \cdots \times p, \\ V_2 &= p \times S_{m_2} \times p \times \cdots \times p, \\ &\cdots\cdots\cdots\cdots \\ V_l &= p \times p \times p \times \cdots \times S_{m_l}. \end{aligned}$$

[33] Man vergl. zum Folgenden: Cartan[3], (b), 25–26.

Dann definieren wir zunächst für

$$i_1 < i_2 < \cdots < i_k$$

das von $V_{i_1}, V_{i_2}, \cdots, V_{i_k}$ aufgespannte Element

$$V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k} = x_1 \times x_2 \times \cdots \times x_l \tag{5.1}$$

durch die Festsetzung:

$$x_j = S_{m_j} \text{ für } j = i_1, i_2, \cdots, i_k,$$

$$x_j = p \quad \text{für alle anderen } j.$$

Durch (5.1) sind 2^l Elemente erklärt, die eine volle additive Basis bilden. Für je zwei Elemente (5.1)

$$X = V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k}, \qquad Y = V_{j_1} \otimes V_{j_2} \otimes \cdots \otimes V_{j_m} \tag{5.2}$$

definieren wir das aufgespannte Element $X \otimes Y$ dadurch, daß wir die rechten Seiten von (5.2) formal nach dem assoziativen Gesetz miteinander multiplizieren und die Regeln

$$V_j \otimes V_i = (-1)^{m_i m_j} V_i \otimes V_j, \qquad V_i \otimes V_i = 0$$

anwenden; dann wird entweder $X \otimes Y = 0$ oder $\pm X \otimes Y$ wieder ein Element (5.1). Schließlich erklären wir für beliebige Elemente X, Y das Produkt $X \otimes Y$ dadurch, daß wir X und Y als lineare Verbindungen mit rationalen Koeffizienten der Elemente (5.1) darstellen und die distributiven Gesetze anwenden.

Durch die damit erklärte Multiplikation wird die additive Gruppe der Homologieklassen des Sphärenproduktes Π zu einem Ring $\mathfrak{Q}(\Pi)$ gemacht. Die Struktur dieses Ringes ist leicht zu übersehen: man stellt leicht fest, daß die Ringe $\mathfrak{R}(\Pi)$ und $\mathfrak{Q}(\Pi)$ miteinander isomorph sind, und daß dieser Isomorphismus durch eine gewisse Dualität vermittelt wird, die sich zunächst darin äußert, daß einem Element x von $\mathfrak{R}(\Pi)$ immer ein Element y von $\mathfrak{Q}(\Pi)$ mit $d(y) = \delta(x)$ entspricht.

Die oben erwähnte *Vermutung* bezüglich des Pontrjaginschen Ringes $\mathfrak{P}(\Gamma)$ einer *Gruppen*-Mannigfaltigkeit Γ ist nun die folgende:

Die, auf Grund des Satzes I mögliche, isomorphe Abbildung der Ringe $\mathfrak{R}(\Gamma)$ *und* $\mathfrak{R}(\Pi)$ *aufeinander läßt sich so wählen, daß sie zugleich die Ringe* $\mathfrak{P}(\Gamma)$ *und* $\mathfrak{Q}(\Pi)$ *isomorph aufeinander abbildet.*

Damit wäre die Struktur des Ringes $\mathfrak{P}(\Gamma)$ sowie die Beziehung zwischen den Ringen $\mathfrak{P}(\Gamma)$ und $\mathfrak{R}(\Gamma)$ weitgehend geklärt.

Unter anderem würde noch die folgende Rolle der minimalen Elemente einer Gruppen-Mannigfaltigkeit sichtbar werden. Offenbar sind die Elemente $V_1, \cdots, V_l$, die durch (**) gegeben sind, minimale Elemente von $\mathfrak{R}(\Pi)$, und zwar bilden sie eine additive Basis der *vollen* Gruppe der minimalen Elemente von $\mathfrak{R}(\Pi)$, d.h. der Vereinigung der Gruppen $\mathfrak{B}_1, \cdots, \mathfrak{B}_n$ (Nr. 33); daher würden auch die Elemente $v_1, \cdots, v_l$ von $\mathfrak{R}(\Gamma)$, welche bei dem vermuteten Isomorphismus den V_i entsprächen, minimal sein und eine Basis der vollen

Gruppe der minimalen Elemente von $\mathfrak{R}(\Gamma)$ bilden; da aber die Elemente (5.1) eine volle additive Basis in Π bilden, würden auch die Pontrjaginschen Produkte

$$v_{i_1} \bigcirc v_{i_2} \bigcirc \cdots \bigcirc v_{i_k},$$
$$i_1 < i_2 \cdots < i_k,$$

eine volle additive Basis in Γ bilden. Es würde sich also herausstellen, daß die minimalen Elemente $v_1, v_2, \cdots, v_l$ in ganz analoger Weise durch die Pontrjaginsche Produktbildung den Ring $\mathfrak{P}(\Gamma)$ aufspannen, wie die maximalen Elemente $z_1, z_2, \cdots, z_l$ durch die Schnittbildung den Ring $\mathfrak{R}(\Gamma)$ erzeugen.[34]

ZÜRICH, SWITZERLAND.

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[34] Die oben ausgesprochene Vermutung, für die ich keinen Beweis gefunden hatte, habe ich mündlich Herrn H. Samelson mitgeteilt, und dieser hat ihre Richtigkeit inzwischen in vollem Umfange bewiesen.—Sie bezieht sich übrigens ausdrücklich auf Gruppenmannigfaltigkeiten, und für den Beweis ist die Benutzung des assoziativen Gesetz wesentlich.

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ON THE CARTAN INVARIANTS OF GROUPS OF FINITE ORDER*

By Richard Brauer

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1. Introduction

E. Cartan, in his fundamental paper on hypercomplex numbers,[1] introduced an important set of invariants $c_{\kappa\lambda}$, $(\kappa, \lambda = 1, 2, \cdots, k)$ of an algebra A with a principal unit 1. Here, k is the number of prime ideals $\mathfrak{P}_\kappa$ of A. The $c_{\kappa\lambda}$ are non-negative integers which also play an important rôle in the decomposition of the regular representation of A.

Let us consider now a semisimple algebra Γ of rank n over an algebraic number field K, and let J be an integral domain[2] of Γ. Every prime ideal $\mathfrak{p}$ of K generates an ideal $\mathfrak{p}_J$ of J. The nature of this ideal is determined by the structure of the residue class ring $J/\mathfrak{p}_J$. This ring can be considered as an algebra over the residue class ring $\mathfrak{o}/\mathfrak{p}$, where $\mathfrak{o}$ denotes the domain of all integers of K. Hence, we may form the Cartan invariants $c_{\kappa\lambda}(\mathfrak{p})$ of $J/\mathfrak{p}_J$. We have in this case $c_{\kappa\lambda}(\mathfrak{p}) = c_{\lambda\kappa}(\mathfrak{p})$, and the $c_{\kappa\lambda}(\mathfrak{p})$ are the coefficients of a non-negative quadratic form $\psi = \sum c_{\kappa\lambda}(\mathfrak{p})x_\kappa x_\lambda$.

In particular, these notions can be used in the case of the group ring Γ of a group $\mathfrak{G}$ of finite order g. As field of reference, we choose an algebraic number field K, such that all the absolutely irreducible representations of $\mathfrak{G}$ can be written with coefficients in K.[3] The linear combinations of the group elements with integral coefficients in K form an integral domain J of Γ. Hence, we may form the Cartan invariants $c_{\kappa\lambda}(\mathfrak{p})$ for every prime ideal $\mathfrak{p}$ of K. It appears that they actually depend only on the rational prime p which is divisible by $\mathfrak{p}$. Accordingly, we denote them by $c_{\kappa\lambda}(p)$.[4] If p is not a divisor of the group order g, then the matrix $C(p) = (c_{\kappa\lambda}(p))$ is the unit matrix 1, i.e. $c_{\kappa\lambda}(p) = \delta_{\kappa\lambda}$. We therefore restrict our attention to the case where p divides g, in which case p is a divisor of the discriminant of J. To every such prime p, we obtain in $c_{\kappa\lambda}(p)$

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[1] E. Cartan, Annales de Toulouse, **12** B, (1898), p. 1. Cf. further R. Brauer, Proc. Nat. Acad. Sci. **25**, (1939), p. 252.

[2] By an integral domain (Ordnung) J of Γ, we understand a subring J of Γ with the following properties: (a) J contains all the integers of K; (b) The rank of J is n; (c) The elements of J, when expressed by a basis $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ of J, have the form $\alpha = \sum \frac{a_i}{w}\epsilon_i$, where the a_i are integers of K and w is a fixed integer of K which is independent of α.

[3] Cf., for instance, A. Speiser, Theorie der Gruppen von endlicher Ordnung, 3 rd ed., Berlin 1937, theorem 181, p. 204.

[4] For the properties of the $c_{\kappa\lambda}(p)$, in this case, cf. R. Brauer and C. Nesbitt, University of Toronto Studies, Math. Series No. 4, 1937, and a paper forthcoming in the Ann. of Math.

a set of invariants of the group $\mathfrak{G}$, which are of great importance for the theory of group characters. The aim of this paper is the determination of the discriminant $|C(p)|$ of ψ. We prove

THEOREM 1: *The determinant* $|c_{\kappa\lambda}(p)|$ *of the matrix of Cartan invariants of a group* $\mathfrak{G}$ *of finite order is a power of* p.

The exact exponent of p in $|c_{\kappa\lambda}(p)|$ is given below in theorem 1*.

2. Preparations for the Proof

We denote by $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_k$, the classes of conjugate elements of the group $\mathfrak{G}$, which consist of p-regular elements.[5] Let $g = g/n_\lambda$ be the number of elements in $\mathfrak{C}_\lambda$, so that n_λ is the order of the normalizor of an element of $\mathfrak{C}_\lambda$. To each class $\mathfrak{C}_\lambda$, there corresponds a reciprocal class $\mathfrak{C}_{\lambda^*}$ containing the reciprocals of the elements of $\mathfrak{C}_\lambda$. We have, then, k absolutely irreducible modular characters, $\varphi^{(1)}, \varphi^{(2)}, \cdots, \varphi^{(k)}$ of $\mathfrak{G}$ (mod p); we may arrange the values $\varphi_\lambda^{(\kappa)}$ of $\varphi^{(\kappa)}$ for the class $\mathfrak{C}_\lambda$ in the form of a matrix

$$\Phi = (\varphi_\lambda^{(\kappa)}) \qquad (\kappa, \lambda = 1, 2, \cdots, k).$$

The number k here also gives the degree of the Cartan matrix $C = (c_{\kappa\lambda}(p))$ and the matrices Φ and C are connected by the formula

$$\Phi' C \Phi = (n_\kappa \delta_{\kappa\lambda^*}),^{6} \tag{1}$$

provided that $\varphi^{(1)}, \varphi^{(2)}, \cdots, \varphi^{(k)}$ are taken in a suitable arrangement. On forming the determinant in (1), we obtain

$$|\Phi|^2 \, |C| = \pm n_1 n_2 \cdots n_k. \tag{2}$$

This shows that $|C| \neq 0$. Hence, the non-negative quadratic form ψ is positive definite, i.e. $|C| > 0$. Further, the determinant $|\Phi|$ is prime to p.[7] Therefore, theorem 1 will be proved when we can show

THEOREM 2: *Let* $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_k$ *be the classes of p-regular, conjugate elements of* $\mathfrak{G}$, *and let* $g_\lambda = g/n_\lambda$ *be the number of elements of* $\mathfrak{C}_\lambda$. *If* Φ *is the matrix of the modular group characters of* $\mathfrak{G}$(mod p), *then the square of the determinant* $|\Phi|$ *is given by*

$$|\Phi|^2 = \pm \frac{n_1 n_2 \cdots n_k}{p^\alpha} \tag{3}$$

where p^α *is the highest power of* p *dividing* $n_1 n_2 \cdots n_k$.

At the same time, we obtain

THEOREM 1*: *The determinant of the Cartan matrix,* $(c_{\kappa\lambda}(p))$, *is equal to* p^α, *where* p^α *has the same significance as in theorem* 2.

For primes p which do not divide the order of $\mathfrak{G}$, theorem 2 is trivial. Here,

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[5] By a p-regular element of $\mathfrak{G}$, we understand an element whose order is prime to p.

[6] Cf. the formulae (29) and (15), respectively, of the papers mentioned in [4].

[7] Cf. the formulae (26), (17), respectively, of the papers mentioned in [4].

k is the number of all the classes of conjugate elements of $\mathfrak{G}$, and $p^{a} = 1$. The relation (3) is obtained by multiplying Φ' and Φ, using the orthogonality relations for group characters. In the same manner, the analogous formula for ordinary group characters (instead of modular characters) can be obtained at once.

In order to prove theorem 2, it is sufficient to show that if $q \neq p$ is a rational prime, and if q^{β} divides the right hand side of (2), then q^{β} divides $|\Phi|^2$. We shall prove that by proving a similar statement for certain minors of Φ. We first have to give some simple group theoretical considerations.

Let A be an element of $\mathfrak{G}$ such that the order of A is prime to p and q. We shall say that an element G of $\mathfrak{G}$ contains A as its *q-regular factor*, if G is of the form $G = AQ$, where the order of Q is a power $q^{\nu} \geqq 1$ of q, and where $AQ = QA$. Of course, A and Q are uniquely determined by G; both can be written as powers of G. If G_1 is conjugate to AQ, then the q-regular factor of G_1 is conjugate to A.

Let $A_1, A_2, \cdots, A_m$ be a maximal system of elements of $\mathfrak{G}$, such that A_i, A_j are not conjugate for $i \neq j$ and the order of each A_i is prime to p and q. With each A_i, we associate those classes of conjugate elements, $\mathfrak{C}_1^{(i)}$, $\mathfrak{C}_2^{(i)}$, $\cdots$, $\mathfrak{C}_{h_i}^{(i)}$, which contain elements with A_i as their q-regular factor. Each of the classes, $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_k$, then appears exactly once in the form $\mathfrak{C}_\mu^{(i)}$. By expanding $|\Phi|$, we see that $|\Phi|$ is a sum of terms

$$T_1 T_2 \cdots T_m, \tag{4}$$

where T_i is a minor of degree h_i of Φ, containing only the columns which belong to $\mathfrak{C}_1^{(i)}, \mathfrak{C}_2^{(i)}, \cdots, \mathfrak{C}_{h_i}^{(i)}$.

We now state

THEOREM 3: *Let A be an element of an order not divisible by the two primes p and q, and assume that the h classes, $\mathfrak{C}_\rho, \mathfrak{C}_\sigma, \cdots, \mathfrak{C}_\tau$, are all the classes of conjugate elements of the group $\mathfrak{G}$, which contain elements G with A as their q-conjugate factor. If $\varphi^{(r)}, \varphi^{(s)}, \cdots, \varphi^{(t)}$ are any h modular characters* (mod p) *of $\mathfrak{G}$, then*

$$|\Delta| = \begin{vmatrix} \varphi_\rho^{(r)} & \varphi_\sigma^{(r)} & \cdots & \varphi_\tau^{(r)} \\ \varphi_\rho^{(s)} & \varphi_\sigma^{(s)} & \cdots & \varphi_\tau^{(s)} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_\rho^{(t)} & \varphi_\sigma^{(t)} & \cdots & \varphi_\tau^{(t)} \end{vmatrix} \equiv 0 \pmod{(q_\rho q_\sigma \cdots q_\tau)^{\frac{1}{2}}}, \tag{5}$$

where q_λ is the highest power of q dividing n_λ.[8]

Each T_i in (4) has the form (5) (for $A = A_i$). From theorem 3, it follows that the expression (4) is divisible by $(q_1 q_2 \cdots q_k)^{\frac{1}{2}}$, and, hence, that $|\Phi|$ is divisible by the same number. It is therefore sufficient to prove theorem 3 in order to prove theorems 1 and 2. Changing the notation, if necessary, we may assume without restriction that

$$\rho = 1, \sigma = 2, \cdots, \tau = h,$$
$$r = 1, s = 2, \cdots, t = h,$$

[8] An analogous theorem holds for ordinary group characters. Here, the assumption that the order of A is prime to p is not necessary. The proof is the same as for theorem 3.

and then (5) assumes the form

$$(6)\qquad |\Delta| = \begin{vmatrix} \varphi_1^{(1)}, & \cdots, & \varphi_h^{(1)} \\ \cdots & \cdots & \cdots \\ \varphi_1^{(h)}, & \cdots, & \varphi_h^{(h)} \end{vmatrix} \equiv 0 \pmod{(q_1 q_2 \cdots q_h)^{\frac{1}{2}}}.$$

The proof of (6) will be given in §3. First, we must formulate and prove a group theoretical lemma. Let $AQ_1, AQ_2, \cdots, AQ_h$ be representatives for the h classes $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_h$, where A is the q-regular factor of AQ_i, and where $Q_1 = 1$. Let $\mathfrak{N}$ be the normalizor of A in $\mathfrak{G}$, and let $\mathfrak{Q}$ be a Sylow subgroup of $\mathfrak{N}$ belonging to the prime q. Then n_1 is the order of $\mathfrak{N}$ and q_1 the order of $\mathfrak{Q}$. Each AQ_i will commute with every element of a certain subgroup $\mathfrak{Q}_i$ of order q_i of $\mathfrak{G}$. Since A and Q_i both are powers of AQ_i, each of them commutes with every element of $\mathfrak{Q}_i$.

In particular, we have $\mathfrak{Q}_i \subseteqq \mathfrak{N}$. Replacing Q_i by an element $N_i^{-1}Q_iN_i$, with N_i in $\mathfrak{N}$, we may assume that

$$(7)\qquad \mathfrak{Q}_i \subseteqq \mathfrak{Q},$$

as follows easily from Sylow's theorem. Since Q_i must belong to $\mathfrak{Q}_i$, the element Q_i itself will belong to $\mathfrak{Q}$. If Q is any element of $\mathfrak{Q}$, then AQ will be conjugate in $\mathfrak{G}$ to some AQ_i, i.e., $G^{-1}AQG = AQ_i$. Raising this equation to suitable exponents, we obtain $G^{-1}AG = A$, $G^{-1}QG = Q_i$. Therefore, Q and Q_i are conjugate in $\mathfrak{N}$, and hence $Q_1, Q_2, \cdots, Q_h$ form a complete system of representatives for those classes of conjugate elements in $\mathfrak{N}$, in which the orders of the elements are powers of q.

In $\mathfrak{Q}$, the elements $Q_1, Q_2, \cdots, Q_h$ need not form a complete system of representatives for the classes of conjugate elements. However, we may construct such a system by adding further elements Q to the set $Q_1, Q_2, \cdots, Q_h$. Each Q will, in $\mathfrak{N}$, be conjugate to a certain Q_i where i is uniquely determined, $i = 1, 2, \cdots, h$. We denote the elements Q belonging to Q_i by $Q_i = Q_i^{(0)}, Q_i^{(1)}, Q_i^{(2)}, \cdots, Q_i^{(l_i)}$, $(l_i \geqq 0)$. Let $q_i^{(\lambda)}$ be the highest power of q dividing the order of the normalizor of $Q_i^{(\lambda)}$ in $\mathfrak{Q}$. According to (7), we have

$$(8)\qquad q_i^{(0)} = q_i.$$

We now prove the

LEMMA: *The numbers $q_i^{(\lambda)}$, $(\lambda = 0, 1, 2, \cdots, l_i)$ are divisors of q_i. If exactly d_i of them are equal to q_i, then*

$$(9)\qquad d_i \not\equiv 0 \pmod{q}.$$

PROOF: Let $\mathfrak{K}_i$ denote the class of conjugate elements of $\mathfrak{N}$, which contains the element AQ_i. The number M_i of elements in $\mathfrak{K}_i$ is equal to the order of $\mathfrak{N}$ divided by the order of the normalizor of AQ_i. Hence

$$(10)\qquad M_i = \frac{q_1}{q_i}\tilde{M}_i, \qquad \text{with } (\tilde{M}_i, q) = 1.$$

The class $\mathfrak{K}_i$ can be broken up into partial classes $\mathfrak{K}_i^{(\mu)}$, where each $\mathfrak{K}_i^{(\mu)}$ consists of elements which are conjugate by means of transformations by elements of $\mathfrak{Q} \subseteq \mathfrak{N}$. The elements $AQ_i^{(0)}, AQ_i^{(1)}, \cdots, AQ_i^{(l_i)}$ will each determine such a partial class, but there may be further partial classes which do not contain elements AQ with Q in $\mathfrak{Q}$. In any case, if $AT_i^{(\mu)}$ is an element of $\mathfrak{K}_i^{(\mu)}$, and if $T_i^{(\mu)}$ commutes with exactly $w_i^{(\mu)}$ elements of $\mathfrak{Q}$, then the number $M_i^{(\mu)}$ of elements of $\mathfrak{K}_i^{(\mu)}$ is given by

$$M_i^{(\mu)} = \frac{q_1}{w_i^{(\mu)}}. \tag{11}$$

We have, of course,

$$M_i = \sum_\mu M_i^{(\mu)}. \tag{12}$$

In $\mathfrak{G}$, the elements AQ_i and $AT_i^{(\mu)}$ are conjugate. Hence, the order of the normalizor of $AT_i^{(\mu)}$ in $\mathfrak{G}$ is divisible by q_i but not by a higher power of q. On considering the subgroup generated by $AT_i^{(\mu)}$ and the $w_i^{(\mu)}$ commuting elements of $\mathfrak{Q}$, we readily see that $w_i^{(\mu)} \leqq q_i$ and that the equality sign can hold only if $T_i^{(\mu)}$ belongs to $\mathfrak{Q}$. When $T_i^{(\mu)} = Q_i^{(\lambda)}$, $w_i^{(\mu)} = q_i^{(\lambda)}$, and we thus obtain the first part of the lemma. If $w_i^{(\mu)} = q_i$, then the partial class $\mathfrak{K}_i^{(\mu)}$ contains exactly one element $Q_i^{(\lambda)}$, $(\lambda = 0, 1, 2, \cdots, l_i)$, and we have $q_i^{(\lambda)} = q_i$. According to our assumption, there are exactly d_i such partial classes. Therefore, d_i of the numbers $M_i^{(\mu)}$, (cf. (11)), are equal to q_1/q_i, the remaining ones being divisible by a higher power power of q. Then (12) gives

$$M_i \equiv d_i \frac{q_1}{q_i} \qquad \left(\operatorname{mod} \frac{qq_1}{q_i}\right)$$

and, on comparing this with (10), we obtain (9).

3. Proof of the Theorems

As we have seen, it is sufficient to prove (6). If $\mathfrak{F}$ is any representation of $\mathfrak{G}$, we may assume that the matrix $\mathfrak{F}_A$ representing the fixed element A appears in canonical form, i.e.

$$A \to \mathfrak{F}_A = \begin{pmatrix} \alpha_1 I_{v_1} & & \\ & \alpha_2 I_{v_2} & \\ & & \ddots \end{pmatrix}$$

where $\alpha_1, \alpha_2, \cdots$ are distinct roots of unity and $v_1, v_2, \cdots$ are positive integers. The matrices $\mathfrak{F}_Q$, representing elements Q of $\mathfrak{Q}$, then break up in the form

$$Q \to \mathfrak{F}_Q = \begin{pmatrix} V_1 & & \\ & V_2 & \\ & & \ddots \end{pmatrix}$$

where V_j is of degree v_j. For a fixed j, the matrices V_j form a representation $\mathfrak{B}_j$ of $\mathfrak{Q}$. Let $\vartheta^{(1)}, \vartheta^{(2)}, \cdots, \vartheta^{(m)}$ be the distinct, irreducible characters[9] of $\mathfrak{Q}$. Then m is the number of classes of conjugate elements in $\mathfrak{Q}$, i.e., the number of elements $Q_i^{(\lambda)}$. If χ denotes the character of $\mathfrak{F}$, we readily obtain

$$\chi(AQ) = \sum_{\mu=1}^{m} z_\mu \vartheta^{(\mu)}(Q)$$

where the z_μ are algebraic integers which are independent of Q. We set, accordingly,

$$\varphi^{(\kappa)}(AQ) = \sum_{\mu} z_{\kappa\mu} \vartheta^{(\mu)}(Q). \tag{13}$$

In particular, we use this for $Q = Q_i^{(\nu)}$. Since Q_i and $Q_i^{(\nu)}$ are conjugate in $\mathfrak{N}$, we have

$$\sum_{\mu} z_{\kappa\mu} \vartheta^{(\mu)}(Q_i) = \sum_{\mu} z_{\kappa\mu} \vartheta^{(\mu)}(Q_i^{(\nu)}). \tag{14}$$

We now introduce matrix notation. Let us denote by Θ the matrix $(\vartheta^{(\kappa)}(Q_i^{(\nu)}))$ of the characters of $\mathfrak{Q}$; the rows here are fixed by one index κ, $(\kappa = 1, 2, \cdots, m)$, the columns by the two indices i, ν, $(i = 1, 2, \cdots, h;\ \nu = 0, 1, 2, \cdots, l_i)$. We arrange the columns so that first the h columns with $\nu = 0$ appear and then the $m - h$ other columns. Thus,

$$\Theta = (\Theta_0, \Theta_1)$$

where Θ_0 is of type (m, h)[10] and Θ_1 of type $(m, m - h)$.

Since the notation in (6) was chosen such that the class $\mathfrak{C}_\lambda$ contained AQ_λ, we have $\varphi_\lambda^{(\kappa)} = \varphi^{(\kappa)}(AQ_\lambda)$. On account of (13), the matrix Δ in (6) can be written in the form

$$\Delta = Z\Theta_0$$

where Z is the matrix $(z_{\kappa\lambda})$, $(\kappa = 1, 2, \cdots, h;\ \lambda = 1, 2, \cdots, m)$, of type (h, m). In order to replace the matrices by square matrices, we set

$$\tilde{Z} = \begin{pmatrix} Z \\ U \end{pmatrix}$$

where U is a matrix of type $(m - h, m)$ with arbitrary integral coefficients. Then

$$\tilde{Z}\Theta = \begin{pmatrix} Z \\ U \end{pmatrix} (\Theta_0, \Theta_1) = \begin{pmatrix} Z\Theta_0 & Z\Theta_1 \\ U\Theta_0 & U\Theta_1 \end{pmatrix} = \begin{pmatrix} \Delta & Z\Theta_1 \\ U\Theta_0 & U\Theta_1 \end{pmatrix}.$$

Here we subtract every column $(i, 0)$ from all the columns (i, λ), with $\lambda > 0$, and with the same first index. Then, according to (14), $Z\Theta_1$ on the right hand

[9] Since the order of $\mathfrak{Q}$ is prime to p, there is no difference here between the ordinary and the modular characters (mod p) of $\mathfrak{Q}$.

[10] i.e., a matrix with m rows and h columns.

side will be replaced by 0. If Θ_1^* is obtained from Θ_1 by subtracting from each column (i, λ), with $\lambda > 0$, the column $(i, 0)$ of Θ_0, then, by taking the determinant, we find

$$|\tilde{Z}||\Theta| = |\Delta||U\Theta_1^*|.$$

Here, $|\tilde{Z}|$ is an algebraic integer, and $|\Theta|$ has the value

$$|\Theta| = \prod_{i=1}^{h}\prod_{\nu=0}^{l_i}(q_i^{(\nu)})^{\frac{1}{2}}.\text{[11]}$$

On account of (8), we obtain

$$|\Delta||U\Theta_1^*| \equiv 0 \pmod{(q_1q_2\cdots q_k)^{\frac{1}{2}}(\textstyle\prod_i\prod_{\lambda>0}q_i^{(\lambda)})^{\frac{1}{2}}}. \tag{15}$$

Let us assume now that the formula (6) does not hold. Since all the $q_i^{(\mu)}$ are powers of q, it follows that

$$|U\Theta_1^*| \equiv 0 \pmod{(q\textstyle\prod_i\prod_{\lambda>0}q_i^{(\lambda)})^{\frac{1}{2}}}, \tag{16}$$

for any choice of U. Taking a suitable U, we see that any minor of degree $m - h$ of Θ_1^* can be obtained in the form $|U\Theta_1^*|$. But the determinant $|(\Theta_1^*)'\Theta_1^*|$ is equal to the sum of the squares of all these minors. Hence,

$$|(\Theta_1^*)'\Theta_1^*| \equiv 0 \pmod{q\textstyle\prod_i\prod_{\lambda>0}q_i^{(\lambda)}}. \tag{17}$$

Any row in $(\Theta_1^*)'$, the transpose of Θ_1^*, is characterized by a pair of indices, i, μ, $(i = 1, 2, \cdots, h; \mu = 1, 2, \cdots, l_i)$, and any column is characterized by an index κ, $(\kappa = 1, 2, \cdots, m)$. The rows of $(\Theta_1^*)'\Theta_1^*$ are given in the same manner, and each column is characterized by a pair of indices j, ν, with $j = 1, 2, \cdots, h$; $\nu = 1, 2, \cdots, l_j$. For the element $y(i, \mu; j, \nu)$ at the place $(i, \mu), (j, \nu)$ in $(\Theta_1^*)'\Theta_1^*$, we obtain easily

$$y(i, \mu; j, \nu) = \sum_{\kappa=1}^{m}(\vartheta^{(\kappa)}(Q_i^{(\mu)}) - \vartheta^{(\kappa)}(Q_i))(\vartheta^{(\kappa)}(Q_j^{(\nu)}) - \vartheta^{(\kappa)}(Q_j))$$

on account of the definition of Θ_1^*. The sum here splits into four sums, each of which can be computed by means of the orthogonality relations for the group characters of $\mathfrak{Q}$. We set $\delta(R, S) = 1$, if the elements R and S^{-1} of $\mathfrak{Q}$ are conjugate in $\mathfrak{Q}$, and in the other case we set $\delta(R, S) = 0$. Since the normalizor of $Q_i^{(\mu)}$ in $\mathfrak{Q}$ has the order $q_i^{(\mu)}$, we find

$$\begin{aligned} &y(i, \mu; j, \nu) \\ &\quad = \delta(Q_i^{(\mu)}, Q_j^{(\nu)})q_i^{(\mu)} - \delta(Q_i, Q_j^{(\nu)})q_i - \delta(Q_i^{(\mu)}, Q_j)q_i^{(\mu)} + \delta(Q_i, Q_j)q_i \end{aligned} \tag{18}$$

For each i, this expression vanishes unless Q_j^{-1} and Q_i are conjugate in $\mathfrak{G}$, since otherwise $Q_i^{(\rho)}$ and $Q_j^{(\sigma)-1}$ could not be conjugate in $\mathfrak{Q}$. Consequently, there is only one value $j = i^*$ for a given i for which the expression can be different

[11] This is the analogue of theorem 2 for ordinary group characters, and, as remarked in connection with theorem 2, this analogue is trivial.

from 0, and we have $l_i = l_{i^*}$. This shows that the determinant (17) splits into a product of m determinants

(19) $$|(\Theta_1^*)'\Theta_1^*| = \pm|\Omega_1|\cdot|\Omega_2|\cdot\cdots\cdot|\Omega_h|,$$

where

(20) $$|\Omega_i| = |y(i,\mu;i^*,\nu)|^{12} \begin{cases} i, i^* \text{ fixed, } \mu \text{ row index,} \\ \nu \text{ column index, } \mu, \nu = 1, 2, \cdots, l_i . \end{cases}$$

According to the lemma in §2, each q_i is divisible by $q_i^{(\mu)}$; we set, accordingly,

(21) $$y(i,\mu,i^*,\nu) = q_i^{(\mu)}x_i(\mu,\nu).$$

Then

(22) $$|\Omega_i| = q_i^{(1)}q_i^{(2)}\cdots q_i^{(l_i)}|X_i|,$$

where

(23) $$X_i = (x_i(\mu,\nu)), \qquad (\mu \text{ row-index, } \nu \text{ column-index}).$$

Using (17), (19), and (22), we obtain

(24) $$|X_1||X_2|\cdots|X_m| \equiv 0 \pmod{q}.$$

Because of (18) and (21), the matrix X_i is the sum of four matrices $X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, X_i^{(4)}$. In each case we have i fixed, and we denote the row-index by μ, the column index by ν.

(25) $$\begin{cases} X_i^{(1)} = (\delta(Q_i^{(\mu)}, Q_{i^*}^{(\nu)}), & X_i^{(2)} = -\left(\delta(Q_i, Q_{i^*}^{(\nu)})\dfrac{q_i^{(\mu)}}{q_i}\right), \\ X_i^{(3)} = -(\delta(Q_i^{(\mu)}, Q_{i^*})), & X_i^{(4)} = \left(\delta(Q_i, Q_{i^*})\dfrac{q_i^{(\mu)}}{q_i}\right). \end{cases}$$

As the lemma in §2 shows, we have $d_i - 1$ values for $\mu \geqq 1$, for which $q_i^{(\mu)} = q_i$. We may assume that these are the values $\mu = 1, 2, \cdots, d_i - 1$.[13] For $\mu \geqq d_i$, we have

(26) $$q_i/q_i^{(\mu)} \equiv 0 \pmod{q}.$$

Three cases must be considered separately.

CASE I: $i \neq i^*$.

We may then assume that $Q_i^{(\mu)-1} = Q_{i^*}^{(\mu)}$ for all μ. Then $X_i^{(1)}$ is the unit matrix, $X_i^{(2)} = X_i^{(3)} = 0$, whereas in $X_i^{(4)}$ each coefficient in the μ-th row is equal to $q_i/q_i^{(\mu)}$. Hence, (mod q), the matrix $X_i^{(4)}$ has $d_i - 1$ rows consisting of 1, and the other rows are all 0. Then

$$X_i^{(4)\prime}X_i^{(4)} \equiv (d_i - 1)X_i^{(4)}, \qquad \text{tr}(X_i^{(4)}) \equiv d_i - 1 \pmod{q}.$$

[12] If $l_i = 0$, we must set $|\Omega_i| = 1$.

[13] For $d_i = 1$, the corresponding kinds of rows of $X_i^{(j)}$, do not occur.

This shows that (mod q) one characteristic root of X has the value $d_i - 1$, and that the others have the value 0. Then the characteristic roots of $X_i = I + X_i^{(4)}$ are given by d_i, 1, 1, $\cdots$, 1 (mod q). Hence

$$|X_i| \equiv d_i \qquad \pmod{q}. \tag{27_1}$$

Case II. $i = i^*$, but Q_i and Q_i^{-1} are not conjugate in $\mathfrak{Q}$.

We may assume here that $Q_i^{-1} = Q_i^{(1)}$. Then $X_i^{(4)} = 0$. In $X_i^{(2)}$, only the first column contains elements different from 0, and the coefficients in this column are given by

$$-q_i/q_i^{(1)}, -q_i/q_i^{(2)}, \cdots, -q_i/q_i^{(l_i)}.$$

In $X_i^{(3)}$, only the first row contains elements different from 0. All the coefficients in the first row are equal to -1.

In $X_i^{(1)}$, the first row and column are 0. Each of the other rows contains exactly one coefficient 1, and all the other coefficients are 0. The same is true for the second, third, $\cdots$, last column. On adding all the other rows to the first row in X_i, we obtain easily

$$|X_i| = \pm\left(\frac{q_i}{q_i^{(1)}} + \frac{q_i}{q_i^{(2)}} + \cdots + \frac{q_i}{q_i^{(l_i)}} + 1\right).$$

Hence, since $d_i - 1$ of the fractions are 1, and the other ones $\equiv 0 \pmod{q}$,

$$|X_i| \equiv \pm d_i \qquad \pmod{q}. \tag{27_2}$$

Case III. $i = i^*$, and Q_i and Q_i^{-1} are conjugate in $\mathfrak{Q}$. Here $X_i^{(2)} = 0$, $X_i^{(3)} = 0$. As in case I, the first $d_i - 1$ rows in $X_i^{(4)}$ contain only coefficients $\equiv 1 \pmod{q}$, and the latter rows contain only coefficients $\equiv 0 \pmod{q}$. The matrix $X_i^{(1)}$ can be changed into the unit matrix, if the columns are taken in another order; the value of $X_i^{(4)} \pmod{q}$ is not altered hereby. The argument used in the first case then gives

$$\pm |X_i| \equiv d_i \qquad \pmod{q}. \tag{27_3}$$

The three formulae (27) show, in connection with the lemma in §2, that in any case $|X_i| \not\equiv 0 \pmod{q}$. Then (24) is impossible. Thus, the assumption that (6) is not true leads to a contradiction, and the theorems 1, 2 and 3 are proved.

University of Toronto

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MINIMAL SURFACES NOT OF MINIMUM TYPE BY A NEW MODE OF APPROXIMATION

MARSTON MORSE AND C. TOMPKINS

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1. Introduction. We begin with a study of a non-negative function $f^0(p)$ defined at each point of a metric space M. We suppose that $f^0(p)$ is the limit of a sequence of functions $f^n(p)$ also defined over M. Let p_n be a homotopic critical point p_n of $f^n(p)$. We study the limit points q of sequences p_n, regarding q as a new type of critical point of the limit function $f^0(p)$.

We give a simple application.

Let g^0 be a simple, closed, rectifiable curve in a euclidean space with the property that the ratio of an arbitrary chord length of g^0 to the smaller of the corresponding two arc lengths of g^0 is bounded from zero. This chord arc condition on g^0 is less restrictive than the condition imposed by Shiffman[1] in his study of unstable minimal surfaces. We make use of Douglas's Dirichlet function $f^0(\varphi)$. The point φ is a monotone transformation of the arc length along g^0. The basic result is as follows.

If g^0 bounds two minimal surfaces of disc type belonging to disjoint minimizing sets of such surfaces (cf. §2), *then g^0 also bounds another minimal surface of disc type not of minimum type.*

Results of this nature but under more restrictive hypotheses have been obtained previously by Morse and Tompkins[2] and by Shiffman, loc. cit. We have found that less restrictive hypotheses on g^0 are adequate, because they can be compensated by more restrictive hypotheses on the approximating functions $f^n(p)$, $n > 0$, making the analysis of the functions $f^n(p)$ much simpler. This applies particularly to the study of upper-reducibility.[3] For it is easy to show that the restricted functions $f^n(p)$ are upper-reducible, while we do not need to show that $f^0(p)$ is upper-reducible.

2. The mode of approximation. Let M be a metric space with points p, q, r and distances pq, pr, etc. satisfying the usual metric axioms. On M there shall be given a sequence of functions $f^n(p)$ converging to a function $f^0(p)$. Thus

$$\lim_{n=\infty} f^n(p) = f^0(p). \tag{2.1}$$

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[1] Shiffman. *The Plateau problem for non-relative minima.* Annals of Mathematics, vol. 40 (1939), pp. 834–854. See (2) §3.

[2] Morse and Tompkins. *The existence of minimal surfaces of general critical types.* Annals of Mathematics, vol. 40 (1939), pp. 443–472. The letters MT will be used in the text to refer to this paper. See corrections elsewhere in this number of the Annals.

[3] Morse. *Functional topology and abstract variational theory.* Mémorial des Sciences Mathématiques, Fascicule 92 (1939). See §8. The letter M will refer to this pamphlet.

This convergence is in general not uniform. The functions $f^m(p)$, $m = 0$, $1, \cdots$, shall satisfy conditions I to VI to be described. If c is an arbitrary constant and f is a function defined on M, f_c shall denote the subset of points of M at which $f \leqq c$. Our first condition is as follows.

HYPOTHESIS I. *Bounded compactness. The sets* f_c^m, $m = 0, 1, \cdots$, *shall be compact for each constant* c.

It follows from this hypothesis that each function f^m is lower semi-continuous.

HYPOTHESIS II. *If* $\lim p_n = p$,

$$\text{inf. lim. } f^n(p_n) \geqq f^0(p). \tag{2.2}$$

HYPOTHESIS III. *Corresponding to any subsequence F^r of the functions f^n and points p_r such that $F^r(p_r)$ is bounded independently of r, there exists a subsequence of the points p_r which converges to a point of M.*

We shall prove the following lemma.

LEMMA 2.1. *Let e and c be positive constants. If n is sufficiently large, and b is a constant sufficiently near c, f_b^n is on the e-neighborhood N_e of f_c^0.*

Suppose the lemma false. There will then exist a constant $e > 0$ and a subsequence F^r of the functions f^n together with points p_r, $r = 1, 2, \cdots$, such that p_r is not on N_e, while

$$c \geqq \sup_{r=\infty} \lim F^r(p_r). \tag{2.3}$$

By virtue of Hypothesis III a subsequence of the points p_r converges to a point q of M.

Without loss of generality we may suppose that $p_1, p_2, \cdots$ is this subsequence. By hypothesis q is not on N_e so that $f^0(q) > c$. But it follows from II that

$$\inf_{r=\infty} \lim F^r(p_r) \geqq f^0(q). \tag{2.4}$$

From (2.3) and (2.4) it follows that $c \geqq f^0(q)$. From this contradiction we infer the truth of the lemma.

HYPOTHESIS IV. *Corresponding to each compact subset A of M, there exists a constant κ^A such that for each integer n, $f^n \leqq \kappa^A f^0$ on A.*

We shall make use of chains and cycles taken in the sense of Vietoris with coefficients in the field of integers mod 2. Cf. M, §1. If p is a point of M, p^* shall represent a 0-cycle each of whose component 0-cycles is on p. This convention will be permanent. The following lemma is an immediate consequence of our definitions.

LEMMA 2.2. *A necessary and sufficient condition that two points p and q belong to the same component of a compact subset A of M is that $p^* + q^* \sim 0$ on A.*

Let f be a function defined on M. A set A will be termed *minimizing* relative to f if f equals a constant c on A, and if there exists an e-neighborhood A_e of A such that $f > C$ on $A_e - A$. If f satisfies Hypothesis I, each minimizing set ω of f is closed. For at a limit point q of ω, $f(q) \leqq c$ by virtue of the lower semi-continuity of f. It follows from the definition of a minimizing set q that q belongs to ω. Hence ω is closed.

A subset of M on which f is bounded will be termed *f-bounded*. We shall prove the following theorem.

THEOREM 2.1. *Let y and z be points respectively of two disjoint minimizing sets ω_1 and ω_2 of f^0. If y and z are connected by an f^0-bounded subset of M and Hypotheses I to IV are satisfied, there exists a subsequence F^r of the functions f^n with the following properties. There exists a least value μ_r of c for which*

$$p^* + q^* \sim 0 \qquad (\text{on } F^r_c),$$

with

$$\mu_r > \max [F^r(y), F^r(z)],$$

while the numbers μ_r converge to a limit μ such that

$$\mu > \max [f^0(y), f^0(z)].$$

We begin the proof of this theorem with certain lemmas. The first lemma concerns a compact subset C of M. A proof may be found in Morse,[4] page 430.

LEMMA 2.3. *If u is a k-cycle on C such that $u \sim 0$ on C_e for each positive e, then $u \sim 0$ on C.*

We shall prove the following lemma.

LEMMA 2.4. *Under the hypotheses of the theorem there exists a least value ν_0 of c such that $y^* + z^* \sim 0$ on f^0_c.*

It follows from the first hypothesis of the theorem and from Lemma 2.2 that there is at least one value of c such that $y^* + z^* \sim 0$ on f^0_c. Let ν_0 be the greatest lower bound of such values of c. There then exist values of c arbitrarily near ν_0 such that $y^* + z^* \sim 0$ on f^0_c. But it follows from Hypothesis I that if c is sufficiently near ν_0, f^0_c lies on an arbitrarily small neighborhood N_e of $f^0_{\nu_0}$. Hence $y^* + z^* \sim 0$ on N_e. We set $C = f^0_{\nu_0}$. We see that $y^* + z^*$ is on C. It follows from Lemma 2.3 that $y^* + z^* \sim 0$ on C, and the proof of Lemma 2.4 is complete.

The points y and z are connected on $f^0_{\nu_0}$. But y and z belong to disjoint minimizing sets. These sets are compact, and hence at a positive distance from each other. It follows that

$$\nu_0 > f^0(y), \qquad \nu_0 > f^0(z). \tag{2.5}$$

The set $f^0_{\nu_0}$ is compact. Hence on this set in accordance with IV, $f^n \leqq \kappa\nu_0$, where κ is a constant. Hence $y^* + z^* \sim 0$ on $f^n_{\kappa\nu_0}$. Let ν_n be the least number c such that $y^* + z^* \sim 0$ on f^n_c. That ν_n exists follows as in the proof of Lemma 2.4. We see that

$$\nu_n \leqq \kappa\nu_0 \qquad (n = 1, 2, \cdots). \tag{2.6}$$

[4] Morse. *Rank and span in functional topology*. Annals of Mathematics, vol. 41 (1940), pp. 419–454.

LEMMA 2.5. *If ν_n is the least number c such that $y^* + z^* \sim 0$ on f_c^n, $n = 0, 1, \cdots$, and if $\mu = \inf \lim_{n=\infty} \nu_n$, then*

$$\mu \geqq \nu_0 . \tag{2.7}$$

Let μ_r be a subsequence of the numbers ν_n such that

$$\lim_{r=\infty} \mu_r = \mu. \tag{2.8}$$

Let F^r be the corresponding subsequence of the functions f^n so that $y^* + z^* \sim 0$ on $F_{\mu_r}^r$. Let e be an arbitrary positive constant, and set $f_\mu^0 = C$. It follows from Lemma 2.1 that $F_{\mu_r}^r \subset C_e$, provided r is sufficiently large. Hence $y^* + z^* \sim 0$ on C_e. It follows from Lemma 2.3 that $y^* + z^* \sim 0$ on C. From the definition of ν_0 we infer that $\mu \geqq \nu_0$, and the lemma is proved.

We shall now prove Theorem 2.1, making use of the sequence F^r and the numbers μ_r of the proof of Lemma 2.5. Observe that

$$\mu > \max [f^0(y), f^0(z)]$$

in accordance with (2.5) and (2.7). Recall that F^r and μ_r converge respectively to f^0 and μ. Hence if ρ is a sufficiently large integer

$$\mu_r > \max [F^r(y), F^r(z)] \qquad (r \geqq \rho).$$

Theorem 2.1 is accordingly satisfied by sequences F^r and μ_r, starting with integers $r \geqq \rho$.

3. h- and fh-critical points. Let F be a function defined on M. We refer to the definition of an F-deformation and a homotopic critical point of F (written h-critical point) as given on page 30 of M. Let $F^1, F^2, \cdots$ be a subsequence of the sequence $f^1, f^2, \cdots$ converging to f^0, and let q_r be an h-critical point of F^r. Any limit point q of the point set q_r will be termed an *fh-critical point* of f^0. This term is relative to the sequence f^n, to which the letter f refers in the term fh-critical point.

LEMMA 3.1. *The set of fh-critical points of f^0 is closed, and the set of h-critical points of f^0 is closed among points at any given level μ.*

That the set of h-critical points of f^0 is closed at the level μ follows at once from the definition of an h-critical point.

To prove that the set of fh-critical points of f^0 is closed, let p be a limit point of a sequence p_r of fh-critical points of f^0. For each fixed r, p_r is by definition the limit of a sequence $p(r, s)$, $s = 1, 2, \cdots$, of h-critical points of functions $f^{n(r,s)}$ where the integer $n(r, s)$ increases with s. Hence if integers $s_1, s_2, \cdots$ are successively chosen with s_r sufficiently large, the sequence $p(r, s_r)$ will converge to p as r becomes infinite, while the integers $n(r, s_r)$ will increase with r. Hence p is an fh-critical point of f^0, and the proof of the lemma is complete.

To continue we need two additional hypotheses. We refer to MT, page 445, for the definition of weak upper-reducibility.

HYPOTHESIS V. *The functions $f^1, f^2, \cdots$ shall be weakly upper-reducible.*

Lemma 8.1 of M clearly holds if weak upper-reducibility replaces upper-reducibility. We shall use Lemma 8.1 of M in proving the following lemma. We refer to Theorem 2.1 and the constants μ, μ_r and functions F^r described therein.

LEMMA 3.2. *The function F^r of Theorem 2.1 assumes the value μ_r in at least one h-critical point w_r such that y, z, w_r are in the same component of $F^r_{\mu_r}$.*

Let κ be the component of $F^r_{\mu_r}$ which contains y and z. The set κ is compact. Suppose there is no h-critical point of $F^r_{\mu_r}$ on κ at the level μ_r. By virtue of Lemma 8.1 of M there will then exist an F^r-deformation of κ into a set definitely below μ_r. Hence $y^* + z^* \sim 0$ on F^r_c for some $c < \mu_r$, contrary to the definition of μ_r.

We infer the truth of the lemma.

HYPOTHESIS VI. (a). *Let p_r be an h-critical point of f^{n_r}, $n_1 < n_2 < \cdots$, such that p_r converges to p as r becomes infinite. Then $f^{n_r}(p_r)$ shall converge to $f^0(p)$.* (b) *Let p_r be an h-critical point of f^0 such that p_r converges to p. Then $f^0(p_r)$ shall converge to $f^0(p)$.*

A point q will be said to be of *non-minimizing* type relative to $f(p)$ if there is a point p in every neighborhood of q such that $f(p) < f(q)$. With this understood we come to a major theorem.

THEOREM 3.1. *If f^0 admits two disjoint minimizing sets ω_1 and ω_2 and if there are points y and z in ω_1 and ω_2 respectively which are connected by an f^0-bounded set, if moreover Hypotheses I to VI are satisfied, then f^0 possesses at least one fh- or h-critical point of non-minimizing type.*

We refer to the numbers μ_r and μ of Theorem 2.1, and prove the following:

(i). *The set σ of h- and fh-critical points of f^0 at the level μ is closed.*

This follows from Lemma 3.1.

(ii). *The set σ is not empty.*

The h-critical point w_r of Lemma 3.2 is a point at which $F^r(w_r) = \mu_r$. These points w_r have at least one cluster point w by virtue of Hypothesis III. It follows from Hypothesis VI (a) that $f^0(w) = \mu$. Thus w is an fh-critical point belonging to σ, so that σ is not empty.

(iii). *There is at least one point of σ of non-minimizing type.*

The set σ, on which $f^0 = \mu$, fails to contain the points y and z of Theorem 2.1, since μ exceeds $f^0(y)$ and $f^0(z)$ as stated in Theorem 2.1. Observe that y, z, and w lie on the same component of f^0_μ since y, z, and w_r lie on the same component of $F^r_{\mu_r}$, and the maximum distance of points of $F^r_{\mu_r}$ from f^0_μ tends to zero as μ_r tends to μ. The set σ cannot be a minimizing set, otherwise σ would be at a positive distance from its complement on f^0_μ, contrary to the fact that y, z, and w are connected on f^0_μ, while y and z are not in σ and w is in σ. Since σ is compact and not a minimizing set of f^0, there exists at least one point of σ of non-minimizing type.

The proof of the theorem is complete.

4. The application to minimal surface theory. Let g^0 be a simple, closed, rectifiable curve in a space of coordinates $(x_1, \cdots, x_n)$. We suppose g^0 has the vector form

$$\mathbf{x} = \mathbf{g}^0(s),$$

where s is the arc length along g^0. The curve g^0 is given with a sense and an origin $s = 0$. So given, g^0 will be termed *coordinated*. It will be convenient to suppose that the total length of g^0 is 2π. We shall assume that g^0 satisfies the following condition.

Chord arc hypothesis. Under this hypothesis the ratio of the length of an arbitrary chord of g^0 to the length of the minimum subtended arc of g^0 shall be bounded from zero for all chords of g^0.

Any simple, closed, regular curve of class C^1 satisfies this hypothesis, as does a simple closed curve composed of a finite set of regular arcs of class C^1, provided at each corner p the two tangent rays directed from p never make a null angle. In particular any simple closed polygon is admissible. This condition is less restrictive than the corresponding condition of Shiffman (loc. cit.). For under Shiffman's condition two tangent rays at a corner cannot make angles less than a right angle, while it is easy to show that under Shiffman's condition the chord arc hypothesis is always satisfied.

A sequence h^n of closed, rectifiable, coordinated curves $\mathbf{x} = \mathbf{h}^n(s)$ will be said to *converge in length*[5] to h^0: $\mathbf{x} = \mathbf{h}^0(s)$ if

$$\lim_{n=\infty} \mathbf{h}^n(s) = \mathbf{h}^0(s)$$

uniformly on each bounded interval for s. It is clear that there exists a sequence of simple, closed, coordinated polygons which converge in length to g^0. The corners of these polygons can be "rounded off" to obtain a sequence of simple, regular, coordinated, closed curves g^n of class C^2 converging in length to g^0. Without loss of generality we can suppose that the length of each curve g^n is 2π, since if this is not already the case, a magnification from the origin in the ratio ρ_n to 1 will bring this about, where ρ_n must be suitably chosen and so chosen converges to 1. Let $\mathbf{x} = \mathbf{g}^n(s)$ be the vector representation of g^n.

Let $\varphi(\alpha)$ be a continuous non-decreasing function of α such that

$$\varphi(\alpha + 2\pi) \equiv \varphi(\alpha) + 2\pi. \tag{4.1}$$

[5] The coördinated curves h^n converge in length to h^0 if the following three conditions are fulfilled. The sensed curve h^n converges to the sensed curve h^0 according to Fréchet, the length of h^n converges to the length of h^0, and the point $s = 0$ on h^n converges to the point $s = 0$ on h^0. For related, but variant, definitions of convergence in length, see the following three sources. Adams and Lewy. *On convergence in length.* Duke Mathematical Journal, vol. 1 (1935), pp. 19–26. McShane. *Curve space topologies associated with variational problems.* Ann. Scuola Norm. Super. Pisa, (2) vol. 9 (1940), pp. 45–60. Morse. *The calculus of variations in the large.* American Mathematical Society Colloquium Publications (1934), p. 209.

We suppose that $\varphi(\alpha)$ satisfies a *three point condition* defined as follows. The relation $\alpha = \varphi(\alpha)$ shall hold for three given distinct values $\alpha_1, \alpha_2, \alpha_3$, of α on the interval $0 \leqq \alpha < 2\pi$, where $\alpha_1, \alpha_2, \alpha_3$ are independent of φ. A function $\varphi(\alpha)$ of this nature will be termed *admissible*. We shall impose another condition on the functions φ when we define the space M.

We admit representations of g^n of the form

$$\mathbf{x} \equiv \mathbf{g}^n[\varphi(\alpha)] \equiv \mathbf{p}^n(\alpha) \qquad (n = 0, 1, \cdots). \tag{4.2}$$

We shall consider the function $\varphi(\alpha)$ as a point in an abstract metric space in which the distance between two points $\varphi(\alpha)$ and $\psi(\alpha)$ shall be the number

$$\psi\varphi = \max | \varphi(\alpha) - \psi(\alpha) | \qquad (0 \leqq \alpha \leqq 2\pi). \tag{4.3}$$

The function $f^n(\varphi)$ shall be defined as the Douglas function[6]

$$f^n(\varphi) = \frac{1}{16\pi} \iint_\omega \frac{[\mathbf{p}^n(\alpha) - \mathbf{p}^n(\beta)]^2}{\sin^2 \dfrac{(\alpha - \beta)}{2}} \, d\alpha \, d\beta, \quad (n = 0, 1, \cdots), \tag{4.4}$$

where ω denotes the parallelogram

$$0 \leqq \beta \leqq 2\pi,$$
$$\beta - \pi \leqq \alpha \leqq \beta + \pi.$$

The Douglas function is an improper integral with an integrand which is singular when $\alpha = \beta$.

As in MT we introduce the improper integral

$$H(\varphi) = \frac{1}{16\pi} \iint_\omega \frac{[\varphi(\alpha) - \varphi(\beta)]^2}{\sin^2 \dfrac{(\alpha - \beta)}{2}} \, d\alpha \, d\beta. \tag{4.5}$$

It follows from the chord arc hypothesis on g^0 and the ordinary properties of length that there exists a constant κ such that

$$[\mathbf{p}^0(\alpha) - \mathbf{p}^0(\beta)]^2 \leqq [\varphi(\alpha) - \varphi(\beta)]^2 \leqq \kappa[\mathbf{p}^0(\alpha) - \mathbf{p}^0(\beta)]^2, \tag{4.6}$$

where κ is independent of the choice of φ. From (4.6) we see that

$$f^0(\varphi) \leqq H(\varphi) \leqq \kappa f^0(\varphi). \tag{4.7}$$

Upon comparing $f^n(\varphi)$ with $H(\varphi)$ we find that

$$f^n(\varphi) \leqq H(\varphi) \leqq \kappa f^0(\varphi). \tag{4.8}$$

Moreover g^n is regular and of class C^2 from which we infer that the ratio of an arbitrary chord length of g^n to either of the corresponding arc lengths exceeds

[6] Douglas. I. *Solution of the problem of Plateau.* Transactions of the American Mathematical Society, vol. 33 (1931), pp. 263–321. II. *The mapping theorem of Koebe and the problem of Plateau.* Journal of Mathematics and Physics, vol. 10 (1931), pp. 106–130.

some positive constant κ_n. It follows as in the proof of (4.8) that $f^0 \leqq \kappa_n f^n$. From this result and from (4.8) we conclude that $f^n(\varphi)$ is finite if and only if $f^0(\varphi)$ is finite.

We shall restrict the space M of points φ to points φ for which $f^0(\varphi)$ is finite.

We shall prove the following theorem.

THEOREM 4.1. *At each point of M,*

$$\lim_{n=\infty} f^n(\varphi) = f^0(\varphi). \tag{4.9}$$

Let e be a positive constant less than π and let ω_e be the subset of the parallelogram ω on which $| \alpha - \beta | < e$. We set

$$\omega = \omega_e + \omega_e^*, \qquad (\omega_e \cdot \omega_e^*) = 0,$$

and

$$f^n(\varphi) = F(\varphi, n, e) + F^*(\varphi, n, e), \qquad (n = 0, 1, \cdots), \tag{4.10}$$

where $F(\varphi, n, e)$ and $F^*(\varphi, n, e)$ are defined as is $f^n(\varphi)$ except that ω_e and ω_e^* shall replace ω as the respective domains of integration. From (4.6) we see that

$$F(\varphi, n, e) \leqq \kappa F(\varphi, 0, e). \tag{4.11}$$

By hypothesis $f^0(\varphi)$ is convergent so that $F(\varphi, 0, e)$ is convergent. Noting that the integral $F^*(\varphi, n, e)$ is proper, we see that

$$\lim_{n=\infty} F^*(\varphi, n, e) = F^*(\varphi, 0, e). \tag{4.12}$$

To establish the theorem, observe that

$$| f^n(\varphi) - f^0(\varphi) | \leqq | F(\varphi, n, e) - F(\varphi, 0, e) | + | F^*(\varphi, n, e) - F^*(\varphi, 0, e) |. \tag{4.13}$$

Corresponding to an arbitrary positive constant η let $e > 0$ be chosen so small that $| F(\varphi, 0, e) | < \eta$. This is possible since $F(\varphi, 0, e)$ is a convergent integral. Making use of (4.11) we see that

$$| F(\varphi, n, e) - F(\varphi, 0, e) | \leqq \kappa\eta + \eta.$$

With e so chosen, let n be so large that

$$| F^*(\varphi, n, e) - F^*(\varphi, 0, e) | < \eta.$$

Making use of (4.13), we then find that

$$| f^n(\varphi) - f^0(\varphi) | \leqq \kappa\eta + \eta + \eta,$$

and the theorem follows at once.

5. Verification of hypotheses. We shall show that the Douglas functions $f^n(\varphi)$ defined in the preceding section satisfy Hypotheses I to VI of §§1 to 3.

HYPOTHESIS I. That M is boundedly compact relative to $f^n(\varphi)$ follows from the work of Douglas II. See also Rado[7] V, 17, and Courant.[8]

HYPOTHESIS II. To verify this hypothesis we recall the origin of the Douglas function. Let g be a simple, closed, rectifiable, coordinated curve given with a vector representation $\mathbf{x} = \mathbf{g}(s)$ in terms of arc length s. Let $\varphi(\alpha)$ be an admissible function φ. Corresponding to the representation $\mathbf{x} = \mathbf{p}(\alpha) = \mathbf{g}[\varphi(\alpha)]$ there exists a harmonic surface $\mathbf{x} = \mathbf{x}(u, v)$ defined and continuous for $u^2 + v^2 \leqq 1$ and such that

$$\mathbf{x}\,(\cos\theta, \sin\theta) \equiv \mathbf{p}(\theta),$$

where r and θ are polar coordinates in the (u, v)-plane. We shall say that the harmonic surface $\mathbf{x} = \mathbf{x}(u, v)$ is *defined* by the pair (g, φ).

The Douglas function $f(\varphi)$ corresponding to g equals the Dirichlet sum

$$D(g, \varphi) = \frac{1}{2}\iint_R \left[\left(\frac{\partial \mathbf{x}}{\partial u}\right)^2 + \left(\frac{\partial \mathbf{x}}{\partial v}\right)^2\right] du\, dv,$$

where R represents the region $u^2 + v^2 < 1$.

Let φ^n be a sequence of points on M converging to φ^0. Recall that g^n converges in length to g^0. Hypothesis II is satisfied if

$$\inf \lim_{n=\infty} D(g^n, \varphi^n) \geqq D(g^0, \varphi^0). \tag{5.1}$$

Relation (5.1) is a consequence of the lower semi-continuity of $D(g, \varphi)$ of which a conventional proof may be indicated as follows. Let $D(g, \varphi, r)$ denote the integral obtained from $D(g, \varphi)$ upon replacing R by the region $u^2 + v^2 < r < 1$. By definition

$$D(g, \varphi) = \lim_{r=1} D(g, \varphi, r).$$

The lower semi-continuity of $D(g, \varphi)$ follows from the fact that $D(g, \varphi, r)$ is continuous in (g, φ), positive and non-decreasing in r.

Relation (5.1) holds, and Hypothesis II follows.

HYPOTHESIS III. Let $\varphi^r(\alpha)$ be an infinite sequence of points φ on M such that

$$f^{n_r}(\varphi^r) \leqq c \qquad (n_1 < n_2 < \cdots; r = 1, 2, \cdots)$$

for some constant c and suitable choices of the integers n_r. To establish III we must prove that some subsequence of the sequence φ^r converges to a point on M.

According to the theory of bounded monotone functions there exists a sub-

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[7] Radó. *On the problem of Plateau.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin (1933).

[8] Courant. *Plateau's problem and Dirichlet's principle.* Annals of Mathematics, vol. 38 (1937), pp. 679–724.

sequence of the functions φ^r which converges for each value of α to a non-decreasing function $\theta(\alpha)$. The function $\theta(\alpha)$ satisfies the three point condition and the condition

$$\theta(\alpha + 2\pi) \equiv \theta(\alpha) + 2\pi.$$

It remains to show that $\theta(\alpha)$ is continuous.

Whether continuous or not $\theta(\alpha)$ possesses right and left limits a_1 and a_2 at each point α_0. Moreover $| a_1 - a_2 | \neq 2\pi$, since $\theta(\alpha)$ satisfies the three point condition. If $| a_1 - a_2 | \neq 0$, it follows as in the proof of Theorem III of Douglas II, page 113, that

$$\sup \lim_{r=\infty} D(g^{n_r}, \varphi^r) = \infty.$$

But

$$f^{n_r}(\varphi^r) = D(g^{n_r}, \varphi^r) \leqq c.$$

From this contradiction we infer that $a_1 = a_2$ so that $\theta(\alpha)$ is continuous. Thus $\theta(\alpha)$ defines a point of M, and Hypothesis III is verified.

HYPOTHESIS IV. This hypothesis is an immediate consequence of (4.8).

HYPOTHESIS V. The functions $f^n(\varphi)$ are weakly upper-reducible for $n > 0$ in accordance with Theorem 5.1 of MT.

The proof of Hypothesis VI is more involved.

HYPOTHESIS VI. Let g be a simple, closed, coordinated, rectifiable curve, and $\varphi(\alpha)$ a transformation which is admissible in the sense of §4. Let $A(g, \varphi)$ be the area of the harmonic surface defined by the pair (g, φ). In a forthcoming paper we shall prove the following theorem of use in establishing VI.

(i) *The area $A(g, \varphi)$ is continuous in its arguments.*

The notion of continuity of $A(g, \varphi)$ is relative to the concept of convergence of g and φ. Convergence of curves g shall be convergence in length, and convergence of φ, convergence defined by the distance function $\varphi\psi$ of (4.3).

A pair (g, φ) which defines a minimal surface will be called *differentially critical.* When (g, φ) is differentially critical,

$$D(g, \varphi) = A(g, \varphi), \tag{5.2}$$

as is well-known. Cf. Douglas II, Theorem I. When φ is an h-critical point of the function $f(\varphi) = D(g, \varphi)$, (g, φ) will be termed *h-critical.* That an h-critical pair is a differentially critical pair follows from Theorem 6.2 of MT. Hence (5.2) holds for h-critical pairs. From (i) we draw the following conclusion.

(ii). *The function defined by $D(g, \varphi)$ on the subset H of h-critical pairs (g, φ) is continuous on H.*

Hypothesis VI (b) is satisfied by virtue of (ii). To verify Hypothesis VI (a) we return to the sequence $f^n(\varphi)$, and prove the following lemma.

LEMMA 5.1. *If ψ^0 is an fh-critical point of f^0, the harmonic surface defined by (g^0, ψ^0) is minimal.*

By definition of an fh-critical point of f^0, ψ^0 is the limit of a sequence of h-critical points ψ^r belonging respectively to a subsequence f^{n_r} of the function f^n. Let

$$S^r: \mathbf{x} = \mathbf{x}^r(u, v) \qquad (r = 1, 2, \cdots)$$

be the minimal surface defined by (g^{n_r}, ψ^r), and let

$$S^0: \mathbf{x} = \mathbf{x}^0(u, v)$$

be the harmonic surface defined by (g^0, ψ^0). The harmonic surface S^0 will be minimal if its differential coefficients satisfy the conditions

$$E(u, v) \equiv G(u, v), \qquad F(u, v) \equiv 0 \qquad (u^2 + v^2 < 1). \tag{5.3}$$

Let E^r, F^r, G^r denote the corresponding differential coefficients of S^r. Recall that

$$\lim_{r=\infty} (g^{n_r}, \psi^r) = (g^0, \psi^0).$$

Upon representing $\mathbf{x}^r(u, v)$ by means of the Poisson integral, one finds that E^r, F^r, G^r converge to E, F, G as r becomes infinite. But S^r is minimal, so that conditions of the form (5.3) hold for S^r. It follows that (5.3) holds for S^0, so that S^0 is minimal.

The proof of the lemma is complete.

By virtue of the lemma, (5.2) holds on the set of fh-critical points of f^0. Hypothesis VI (a) is satisfied by virtue of (i).

Our principal theorem is as follows.

THEOREM 5.1. *Let g^0 be a simple, closed, rectifiable curve g^0 which satisfies the chord arc hypothesis. If g^0 bounds two minimal surfaces of disc type defined respectively by points on disjoint minimizing sets of the Douglas function $f^0(\varphi)$, then g^0 also bounds a minimal surface of disc type but not of minimum type.*

It follows from the hypotheses of the theorem that $f^0(\varphi)$ possesses two disjoint minimizing subsets ω_1 and ω_2. Let φ and ψ be points on ω_1 and ω_2 respectively. The 1-parameter family of points

$$t\psi(\alpha) + (1 - t)\varphi(\alpha), \qquad (0 \leqq t \leqq 1),$$

is an f^0-bounded arc connecting φ with ψ as has been shown in MT, page 451. The first hypothesis of Theorem 3.1 is accordingly satisfied.

Hypotheses I to VI hold for the sequence $f^n(\varphi)$ as we have seen. Theorem 3.1 thus applies, and we infer the existence of at least one fh- or h-critical point ψ^0 of f^0 not of minimum type. But we have just seen that such a point ψ^0 defines a minimal surface of disc type bounded by g^0; this surface is not of minimum type relative to the Douglas function f^0.

The proof of the theorem is complete.

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ON A LEMMA OF McSHANE[1]

BY TIBOR RADÓ

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1. McShane, in his work on the lower semi-continuity of double integrals in Calculus of Variations,[2] established the following important lemma.[3] In the unit square Q: $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, let there be given triples of continuous functions $x(u, v)$, $y(u, v)$, $z(u, v)$, $x_n(u, v)$, $y_n(u, v)$, $z_n(u, v)$ such that the following conditions are satisfied.

I. The partial derivatives x_u, x_v, y_u, y_v, z_u, z_v exist almost everywhere in Q.

II. The Jacobians

$$X = y_u z_v - y_v z_u, \qquad Y = z_u x_v - z_v x_u, \qquad Z = x_u y_v - x_v y_u$$

are summable in Q.

III. The functions $x(u, v)$, $y(u, v)$, $z(u, v)$ are absolutely continuous on the boundary B of Q.

IV. We have

$$\iint_Q X\,du\,dv = \frac{1}{2}\int_B (y\,dz - z\,dy), \qquad \iint_Q Y\,du\,dv = \frac{1}{2}\int_B (z\,dx - x\,dz),$$

$$\iint_Q Z\,du\,dv = \frac{1}{2}\int_B (x\,dy - y\,dx).$$

V. The functions $x_n(u, v)$, $y_n(u, v)$, $z_n(u, v)$ are quasi-linear[4] in Q.

VI. $x_n(u, v) \to x(u, v)$, $y_n(u, v) \to y(u, v)$, $z_n(u, v) \to z(u, v)$ uniformly in Q.

Then for every choice of the constants a, b, c there exists a sequence $\{V_n\}$ *of measurable subsets of Q such that*

$$\iint_{V_n} (aX_n + bY_n + cZ_n)\,du\,dv \xrightarrow[n\to\infty]{} \iint_Q (aX + bY + cZ)\,du\,dv,$$

where

$$X_n = y_{nu} z_{nv} - y_{nv} z_{nu}, \quad Y_n = z_{nu} x_{nv} - z_{nv} x_{nu}, \quad Z_n = x_{nu} y_{nv} - x_{nv} y_{nu}.$$

2. In view of the importance of this lemma, it seemed to be of interest to investigate the possibility of replacing the conditions I–IV of McShane, con-

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[1] Presented to the American Mathematical Society at the meeting in Chicago, April 1940.

[2] E. J. McShane, *Integrals over surfaces in parametric form*, Annals of Mathematics, vol. 34, 1933, pp. 815–838.

[3] Loc. cit.,[2] p. 829.

[4] A continuous function $f(u, v)$ is quasi-linear in Q if Q can be subdivided into a finite number of triangles on each of which $f(u, v)$ is linear.

cerned with the limit triple $x(u, v)$, $y(u, v)$, $z(u, v)$, by less restrictive ones. It should be noted that the purpose of conditions III and IV is to provide, for the integral of $aX + bY + cZ$, a geometrical interpretation which is essential for the proof of McShane. It may be therefore of interest to point out that *the lemma remains valid if the apparently decisive conditions* III *and* IV *are dropped.* The purpose of this paper is to prove this assertion. Applications of the result will be considered on another occasion.

3. As stated above, the lemma involves three arbitrary constants a, b, c, but it is evident from the work of McShane that it is sufficient to establish the apparently very special case $a = b = 0$, $c = 1$, since the general case can then be disposed of by a very simple device. For this reason, and also for easier reference, we wish to state our result for this special case explicitly.

4. THEOREM. *In the unit square* Q: $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$, *let there be given pairs of functions* $x(u, v)$, $y(u, v)$, $x_n(u, v)$, $y_n(u, v)$ *such that the following conditions are satisfied.*

I. *The partial derivatives* x_u, x_v, y_u, y_v *exist almost everywhere in* Q.

II. *The Jacobian* $J = x_u y_v - x_v y_u$ *is summable in* Q.

III. *The functions* $x_n(u, v)$, $y_n(u, v)$ *are quasi-linear in* Q.

IV. $x_n(u, v) \to x(u, v)$, $y_n(u, v) \to y(u, v)$ *uniformly in* Q.

Then there exists a sequence of measurable subsets V_n *of* Q *such that*

$$\iint_{V_n} J_n \, du \, dv \to \iint_{Q} J \, du \, dv,$$

where $J_n = x_{nu} y_{nv} - x_{nv} y_{nu}$.

5. Applying a device due to McShane,[5] we proceed as follows to derive from this statement the one described in section 2. Let us consider the triples $x(u, v)$, $y(u, v)$, $z(u, v)$, $x_n(u, v)$, $y_n(u, v)$, $z_n(u, v)$, but let us assume only that conditions I, II, V, VI of section 1 are satisfied. Let there be given a set of constants a, b, c. Without loss of generality we can assume that $a^2 + b^2 + c^2 = 1$. We can find then an orthogonal matrix, with determinant $+1$, of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a & b & c \end{pmatrix}.$$

Let us put

$$\bar{x}(u, v) = a_{11}x(u, v) + a_{12}y(u, v) + a_{13}z(u, v),$$

$$\bar{x}_n(u, v) = a_{11}x_n(u, v) + a_{12}y_n(u, v) + a_{13}z_n(u, v),$$

$$\bar{y}(u, v) = a_{21}x(u, v) + a_{22}y(u, v) + a_{23}z(u, v),$$

$$\bar{y}_n(u, v) = a_{21}x_n(u, v) + a_{22}y_n(u, v) + a_{23}z_n(u, v).$$

Let $\bar{J}$, $\bar{J}_n$ denote the Jacobians of the pairs $\bar{x}$, $\bar{y}$ and $\bar{x}_n$, $\bar{y}_n$ respectively. Clearly, the pairs $\bar{x}$, $\bar{y}$, $\bar{x}_n$, $\bar{y}_n$ satisfy the conditions of section 4 and therefore we have,

[5] See loc. cit.[3]

by the theorem stated in that section, a sequence $\{V_n\}$ of measurable subsets of Q such that

$$\iint_{V_n} \mathfrak{J}_n \, du \, dv \to \iint_{Q} \mathfrak{J} \, du \, dv.$$

This sequence possesses the desired property, since we have by direct computation

$$\mathfrak{J} = aX + bY + cZ, \qquad \mathfrak{J}_n = aX_n + bY_n + cZ_n .$$

6. The rest of this paper contains the proof of the theorem of section 4. Sections 7–12 contain some facts needed in the sequel, while sections 13–17 contain the proof itself.

7. Let G be a bounded measurable set in the xy plane and $h(x, y)$, $h_n(x, y)$ summable functions on G such that $h_n(x, y) \to h(x, y)$ on G. We assume that these functions do not take on the values $\pm\infty$. Under these conditions, there exists a sequence of measurable subsets G_n of G such that

$$\iint_{G_n} h_n \, dx \, dy \to \iint_{G} h \, dx \, dy.$$

PROOF. Define

$$\kappa_n = \text{gr.l.b.} \left| \iint_{G} h \, dx \, dy - \iint_{E} h_n \, dx \, dy \right|,$$

where the greatest lower bound is taken with respect to all measurable subsets E of G. Give any $\epsilon > 0$. Since $h(x, y)$ is summable on G, we have then an $\eta = \eta(\epsilon) > 0$ such that if S is any measurable subset of G we have

$$\left| \iint_{S} h \, dx \, dy \right| < \epsilon \quad \text{if} \quad |S| < \eta.$$ [6]

Since $h_n \to h$ on G, we have by the theorem of Egoroff[7] a measurable subset H of G such that

$$|G - H| < \eta$$

and such that $h_n \to h$ *uniformly* on H. Hence

$$\iint_{H} |h - h_n| \, dx \, dy \to 0 \quad \text{for} \quad n \to \infty,$$

and consequently

$$0 \leqq \overline{\lim_{n\to\infty}} \, \kappa_n \leqq \epsilon,$$

[6] If S is a measurable set, then $|S|$ denotes its Lebesgue measure.

[7] See for instance Saks, *Theory of the integral*, Warszawa 1937, p. 18.

since

$$0 \leqq \kappa_n \leqq \left| \iint_G h\,dx\,dy - \iint_H h_n\,dx\,dy \right|$$

$$\leqq \left| \iint_{G-H} h\,dx\,dy \right| + \iint_H |h - h_n|\,dx\,dy < \epsilon + \iint_H |h - h_n|\,dx\,dy.$$

As $\epsilon > 0$ was arbitrary, it follows that $\kappa_n \to 0$. By the definition of κ_n, we have a measurable subset G_n of G such that

$$\left| \iint_G h\,dx\,dy - \iint_{G_n} h_n\,dx\,dy \right| < \kappa_n + \frac{1}{n}.$$

As $\kappa_n \to 0$, the sequence G_n has therefore the desired property.

8. Suppose the functions $f(u, v)$, $g(u, v)$ are merely continuous on the unit square Q: $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$. Let (u_0, v_0) be an interior point of Q and q a square in Q with center at (u_0, v_0) and sides parallel to the axes. Denote by b the boundary of q. Take four constants $\alpha, \beta, \gamma, \delta$ and put

$$\rho(u_0, v_0, q, f, \alpha, \beta) = \max_{(u,v)\epsilon b} \frac{|f(u, v) - f(u_0, v_0) - \alpha(u - u_0) - \beta(v - v_0)|}{[(u - u_0)^2 + (v - v_0)^2]^{\frac{1}{2}}},$$

$$\rho(u_0, v_0, q, g, \gamma, \delta) = \max_{(u,v)\epsilon b} \frac{|g(u, v) - g(u_0, v_0) - \gamma(u - u_0) - \delta(v - v_0)|}{[(u - u_0)^2 + (v - v_0)^2]^{\frac{1}{2}}}.$$

If there exists *some* sequence of squares q_n such that simultaneously

$$\rho(u_0, v_0, q_n, f, \alpha, \beta) \to 0, \quad \rho(u_0, v_0, q_n, g, \gamma, \delta) \to 0, \quad |q_n| \to 0, \tag{1}$$

then we shall say that the pair $f(u, v)$, $g(u, v)$ satisfies condition C at the point (u_0, v_0) with the constants $\alpha, \beta, \gamma, \delta$.

9. The reader will have noticed that the preceding condition C is merely a greatly weakened form of *total differentiability*. As it might be expected, the methods developed by Rademacher, Stepanoff and others to obtain results of surprising generality concerning total differentiability,[8] can be easily adapted to our condition C. One statement obtained in this manner is as follows.[9] Suppose that the functions $f(u, v)$, $g(u, v)$ are continuous in Q and that the partial derivatives f_u, f_v, g_u, g_v exist almost everywhere in Q. Then condition C is satisfied at almost every point (u_0, v_0) of Q with the constants $\alpha = f_u(u_0, v_0)$, $\beta = f_v(u_0, v_0)$, $\gamma = g_u(u_0, v_0)$, $\delta = g_v(u_0, v_0)$.

10. Consider again a pair of functions $f(u, v)$, $g(u, v)$ which we assume to be merely continuous in Q. These functions define a continuous transformation

[8] See the excellent presentation loc. cit.,[7] Ch. 9.

[9] For the details of the proof, see the author's paper, *On absolutely continuous transformations in the plane*, Duke Math. Journal, vol. 4, 1938, pp. 189–221, in particular p. 219.

$$T: \begin{cases} x = f(u, v) \\ y = g(u, v) \end{cases} \qquad (u, v) \in Q.$$

Let q be a square in Q, with sides parallel to the axes, and let b be the boundary of q. If the point (u, v) describes b in the counter-clockwise sense, its image under T describes in the xy-plane a continuous closed oriented curve b'. In the xy-plane we define a function $\mu(x, y, q, T)$ as follows. If the point (x, y) is on b', then $\mu(x, y, q, T) = 0$. Otherwise $\mu(x, y, q, T)$ is equal to the topological index of (x, y) with respect to b'. We shall denote by $S_1(q, T)$, $S_2(q, T)$ the point-sets in the xy-plane where $\mu(x, y, q, T)$ is equal to $+1$ and -1 respectively. These sets are bounded and open (possibly vacuous).

11. Keeping the notations of the preceding section, let us consider an interior point (u_0, v_0) of Q where the condition C of section 8 is satisfied with some constants $\alpha, \beta, \gamma, \delta$. Let $\{q_n\}$ be a sequence of squares, as specified in section 8, such that the relations (1) hold. We consider the auxiliary transformation

$$T^*: \begin{cases} x = f(u_0, v_0) + \alpha(u - u_0) + \beta(v - v_0), \\ y = g(u_0, v_0) + \gamma(u - u_0) + \delta(v - v_0), \end{cases} \qquad (u, v) \in Q.$$

Assuming that $\alpha\delta - \beta\gamma \neq 0$, the image of the square q_n under T^* is a parallelogram π_n^* in the xy-plane. Suppose first that $\alpha\delta - \beta\gamma > 0$. Then the set $S_1(q_n, T^*)$ is simply the interior of π_n^*, and condition C implies, by a familiar elementary reasoning,[10] that for large values of n the symmetric difference[11] of the sets $S_1(q_n, T^*)$ and $S_1(q_n, T)$ is contained in a narrow strip σ_n, bounded by two parallelograms similar to π_n^*, such that $|\sigma_n|/|\pi_n^*| \to 0$. As $|\pi_n^*|/|q_n| \to \alpha\delta - \beta\gamma$, we also have $|\sigma_n|/|q_n| \to 0$. Thus a fortiori

$$\frac{|S_1(q_n, T)| - |S_1(q_n, T^*)|}{|q_n|} \to 0.$$

But, since T^* is an affine transformation,

$$\frac{|S_1(q_n, T^*)|}{|q_n|} = \frac{|\pi_n^*|}{|q_n|} \to \alpha\delta - \beta\gamma.$$

Hence

$$\frac{|S_1(q_n, T)|}{|q_n|} \to \alpha\delta - \beta\gamma \quad \text{if} \quad \alpha\delta - \beta\gamma > 0.$$

Similarly it follows that

$$-\frac{|S_2(q_n, T)|}{|q_n|} \to \alpha\delta - \beta\gamma \quad \text{if} \quad \alpha\delta - \beta\gamma < 0.$$

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[10] For a detailed discussion of a practically identical situation, see the author's paper, *Über das Flächenmass rektifizierbarer Flächen*, Math. Annalen, vol. 100, 1928, pp. 445–479, in particular pp. 461–466.

[11] The symmetric difference of two sets A and B consists of those points which belong to exactly one of A, B.

12. Using the notations of section 10, let us assume that $f(u, v)$, $g(u, v)$ are quasi-linear in Q. Let us denote by $J(u, v)$ the Jacobian $f_u g_v - f_v g_u$. Let us take, in the xy-plane, any summable function $F(x, y)$ which takes on only finite values (summability meaning that $F(x, y)$ is summable on every bounded measurable set). Then we have for every square q in Q the transformation formula

$$\iint_q F[f(u, v,), g(u, v)]J(u, v)\,du\,dv = \iint F(x, y)\mu(x, y, q, T)\,dx\,dy.^{12}$$

As a matter of fact, this formula holds under very general conditions.[13] For quasi-linear transformations, of course, the formula is practically trivial.

13. We proceed presently to prove the theorem of section 4. Through the rest of the paper the notations of that section will be used. Let q be a square in Q with sides parallel to the axes. We define

$$\lambda_n(q) = \underset{E \subset q}{\text{gr.l.b.}} \left| \iint_q J\,du\,dv - \iint_E J_n\,du\,dv \right|;$$

that is, the greatest lower bound is taken with respect to all measurable subsets E of q. We define further

$$\lambda(q) = \overline{\lim_{n \to \infty}}\, \lambda_n(q).$$

By taking in q a set E of measure zero, we see that

$$\lambda_n(q) \leqq \left| \iint_q J\,du\,dv \right| \leqq \iint_q |J|\,du\,dv, \tag{2}$$

and hence also

$$\lambda(q) \leqq \iint |J|\,du\,dv.$$

Take now in Q any finite or infinite sequence of non-overlapping squares $q_1, q_2, \cdots, q_i, \cdots$, such that

$$|Q - \sum q_i| = 0.$$

We assert that

$$\lambda(Q) \leqq \sum \lambda(q_i).$$

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[12] As $\mu(x, y, q, T)$ vanishes outside of a sufficiently large circle, the range of integration on the right can be taken as the whole xy-plane.

[13] See, also for further literature, loc. cit.[9] It is worth noting that the quasi-linear character of $f(u, v)$, $g(u, v)$ is used only to secure the above transformation formula. This remark suggests further generalizations which we do not wish to discuss at this time.

PROOF.[14] Observe first that the series on the right is convergent since it is dominated by the series

$$\sum \iint_{q_j} |J| \, du \, dv,$$

which converges because the squares q_j do not overlap. Give now $\epsilon > 0$. By the definition of $\lambda_n(q_j)$, we have a measurable subset E_n^j of q_j such that

$$\left| \iint_{q_j} J \, du \, dv - \iint_{E_n^j} J_n \, du \, dv \right| < \lambda_n(q_j) + \frac{\epsilon}{2^j}. \tag{3}$$

Put

$$E_n = \sum_j E_n^j.$$

Since the squares q_j do not overlap, we have by (3)

$$\lambda_n(Q) \leqq \left| \iint_Q J \, du \, dv - \iint_{E_n} J_n \, du \, dv \right|$$

$$\leqq \sum_j \left| \iint_{q_j} J \, du \, dv - \iint_{E_n^j} J_n \, du \, dv \right| < \sum_j \lambda_n(q_j) + \epsilon. \tag{4}$$

Take any positive integer i. We have then by (2)

$$\sum_j \lambda_n(q_j) \leqq \sum_{j=1}^{i} \lambda_n(q_j) + \sum_{j=i+1}^{\infty} \iint_{q_j} |J| \, du \, dv.$$

Hence

$$\overline{\lim_{n\to\infty}} \sum_j \lambda_n(q_j) \leqq \sum_{j=1}^{i} \lambda(q_j) + \sum_{j=i+1}^{\infty} \iint_{q_j} |J| \, du \, dv.$$

Since the squares q_j do not overlap, the second summation on the right converges to zero for $i \to \infty$. Thus

$$\overline{\lim_{n\to\infty}} \sum_j \lambda_n(q_j) \leqq \sum_j \lambda(q_j).$$

From (4) we infer now

$$\lambda(Q) = \overline{\lim_{n\to\infty}} \lambda_n(Q) \leqq \sum \lambda(q_j) + \epsilon.$$

Since ϵ was arbitrary, this proves our assertion.

14. We consider now the transformation

$$T: \begin{cases} x = x(u, v), \\ y = y(u, v), \end{cases} \qquad (u, v) \in Q.$$

[14] We discuss the case of an infinite sequence q_j, since the finite case is trivial.

12. Using the notations of section 10, let us assume that $f(u, v)$, $g(u, v)$ are quasi-linear in Q. Let us denote by $J(u, v)$ the Jacobian $f_u g_v - f_v g_u$. Let us take, in the xy-plane, any summable function $F(x, y)$ which takes on only finite values (summability meaning that $F(x, y)$ is summable on every bounded measurable set). Then we have for every square q in Q the transformation formula

$$\iint_q F[f(u, v,), g(u, v)] J(u, v)\, du\, dv = \iint F(x, y)\mu(x, y, q, T)\, dx\, dy.$$[12]

As a matter of fact, this formula holds under very general conditions.[13] For quasi-linear transformations, of course, the formula is practically trivial.

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that is, the greatest lower bound is taken with respect to all measurable subsets E of q. We define further

$$\lambda(q) = \overline{\lim_{n \to \infty}}\, \lambda_n(q).$$

By taking in q a set E of measure zero, we see that

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and hence also

$$\lambda(q) \leqq \iint |J|\, du\, dv.$$

Take now in Q any finite or infinite sequence of non-overlapping squares $q_1, q_2, \cdots, q_j, \cdots$, such that

$$|Q - \sum q_j| = 0.$$

We assert that

$$\lambda(Q) \leqq \sum \lambda(q_j).$$

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[12] As $\mu(x, y, q, T)$ vanishes outside of a sufficiently large circle, the range of integration on the right can be taken as the whole xy-plane.

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PROOF.[14] Observe first that the series on the right is convergent since it is dominated by the series

$$\sum \iint_{q_i} |J| \, du \, dv,$$

which converges because the squares q_i do not overlap. Give now $\epsilon > 0$. By the definition of $\lambda_n(q_i)$, we have a measurable subset E_n^i of q_i such that

$$\left| \iint_{q_i} J \, du \, dv - \iint_{E_n^i} J_n \, du \, dv \right| < \lambda_n(q_i) + \frac{\epsilon}{2^i}. \tag{3}$$

Put

$$E_n = \sum_i E_n^i.$$

Since the squares q_i do not overlap, we have by (3)

$$\lambda_n(Q) \leqq \left| \iint_Q J \, du \, dv - \iint_{E_n} J_n \, du \, dv \right|$$

$$\leqq \sum_i \left| \iint_{q_i} J \, du \, dv - \iint_{E_n^i} J_n \, du \, dv \right| < \sum_i \lambda_n(q_i) + \epsilon. \tag{4}$$

Take any positive integer i. We have then by (2)

$$\sum_j \lambda_n(q_j) \leqq \sum_{j=1}^{i} \lambda_n(q_j) + \sum_{j=i+1}^{\infty} \iint_{q_j} |J| \, du \, dv.$$

Hence

$$\overline{\lim_{n\to\infty}} \sum_j \lambda_n(q_j) \leqq \sum_{j=1}^{i} \lambda(q_j) + \sum_{j=i+1}^{\infty} \iint_{q_j} |J| \, du \, dv.$$

Since the squares q_j do not overlap, the second summation on the right converges to zero for $i \to \infty$. Thus

$$\overline{\lim_{n\to\infty}} \sum_j \lambda_n(q_j) \leqq \sum_j \lambda(q_j).$$

From (4) we infer now

$$\lambda(Q) = \overline{\lim_{n\to\infty}} \lambda_n(Q) \leqq \sum \lambda(q_j) + \epsilon.$$

Since ϵ was arbitrary, this proves our assertion.

14. We consider now the transformation

$$T: \begin{cases} x = x(u, v), \\ y = y(u, v), \end{cases} \qquad (u, v) \in Q.$$

[14] We discuss the case of an infinite sequence q_i, since the finite case is trivial.

Let q be a square in Q with sides parallel to the axes. We assert the inequality[15]

$$\lambda(q) \leqq \min \left\{ \left| \iint_q J\, du\, dv - | S_1(q, T) | \right|, \left| \iint_q J\, du\, dv + | S_2(q, T) | \right| \right\}, \tag{5}$$

where the sets $S_1(q, T)$, $S_2(q, T)$ are defined as in section 10.

PROOF. Let us consider the transformations

$$T_n: \begin{cases} x = x_n(u, v), \\ y = y_n(u, v), \end{cases} \qquad (u, v) \in Q.$$

In the xy-plane, let S be the set where $\mu(x, y, q, T) \neq 0$. Since $x_n(u, v) \to x(u, v)$, $y_n(u, v) \to y(u, v)$ uniformly in Q, and hence in q, we have

$$\mu(x, y, q, T_n) \to \mu(x, y, q, T)$$

on S, and hence also on the sets $S_1(q, T)$, $S_2(q, T)$. Since T_n is quasi-linear, $\mu(x, y, q, T_n)$ is summable. We cannot assert that $\mu(x, y, q, T)$ is summable, but this function is surely summable on $S_1(q, T)$ and $S_2(q, T)$, since $| \mu(x, y, q, T) | = 1$ on these sets. Hence we can apply the remark in section 7 with $h(x, y) = \mu(x, y, q, T)$, $h_n(x, y) = \mu(x, y, q, T_n)$ and first with $G = S_1(q, T)$ and second with $G = S_2(q, T)$. It follows that we have sequences of bounded measurable sets G_n', G_n'' such that

$$\iint_{G_n'} \mu(x, y, q, T_n)\, dx\, dy \to \iint_{S_1(q,T)} \mu(x, y, q, T)\, dx\, dy = | S_1(q, T) |, \tag{6}$$

$$\iint_{G_n''} \mu(x, y, q, T_n)\, dx\, dy \to \iint_{S_2(q,T)} \mu(x, y, q, T)\, dx\, dy = -| S_2(q, T) |.$$

Now let E_n' be the complete model[16] in q of the set G_n' under the transformation T_n. The transformation formula of section 12 yields then, if $F(x, y)$ is taken as the characteristic function of the set G_n', the formula

$$\iint_{G_n'} \mu(x, y, q, T_n)\, dx\, dy = \iint_{E_n'} J_n\, du\, dv. \tag{7}$$

Similarly, if E_n'' is the complete model of G_n'' in q under the transformation T_n, we have

$$\iint_{G_n''} \mu(x, y, q, T_n)\, dx\, dy = \iint_{E_n''} J_n\, du\, dv.$$

Now, by the definition of $\lambda_n(q)$,

$$\lambda_n(q) \leqq \left| \iint_q J\, du\, dv - \iint_{E_n'} J_n\, du\, dv \right|,$$

---

[15] If a, b are real numbers, then min (a, b) denotes the smaller one of a, b if $a \neq b$ and the common value of a, b if $a = b$.

[16] That is, E_n' is the set of all those points in q whose image under T_n is comprised in G_n'.

and for $n \to \infty$ it follows by (6), (7) that

$$\lambda(q) \leqq \left| \iint_q J\,du\,dv - |S_1(q, T)| \right|.$$

Similarly

$$\lambda(q) \leqq \left| \iint_q J\,du\,dv + |S_2(q, T)| \right|.$$

The last two inequalities imply (5).

15. Denote now by E the set of those interior points (u_0, v_0) of Q where the following conditions are satisfied.

I. x_u, x_v, y_u, y_v exist at (u_0, v_0).

II. Condition C of section 8 is satisfied at (u_0, v_0) with the constants $\alpha = x_u(u_0, v_0)$, $\beta = x_v(u_0, v_0)$, $\gamma = y_u(u_0, v_0)$, $\delta = y_v(u_0, v_0)$.

III. If q is a square with sides parallel to the axes and with center at (u_0, v_0), then

$$\frac{\iint_q J\,du\,dv}{|q|} \to J(u_0, v_0) \quad \text{for} \quad |q| \to 0.$$

Since $J(u, v)$ exists almost everywhere in Q and is summable there by assumption, III is satisfied almost everywhere in Q by a well-known theorem, while II is satisfied almost everywhere in Q by section 9. Thus

$$|Q - E| = 0.$$

16. Now let (u_0, v_0) be a point of the set E of the preceding section. By condition II, we have then a sequence $q_n(u_0, v_0)$ of squares with sides parallel to the axes and with center at (u_0, v_0) for which the relations (1) of section 8 hold with $f(u, v) = x(u, v)$, $g(u, v) = y(u, v)$, $\alpha = x_u(u_0, v_0)$, $\beta = x_v(u_0, v_0)$, $\gamma = y_u(u_0, v_0)$, $\delta = y_v(u_0, v_0)$. By section 11 we have therefore

$$\frac{|S_1(q_n(u_0, v_0), T)|}{|q_n(u_0, v_0)|} \to J(u_0, v_0) \quad \text{if} \quad J(u_0, v_0) > 0,$$

and

$$-\frac{|S_2(q_n(u_0, v_0), T)|}{|q_n(u_0, v_0)|} \to J(u_0, v_0) \quad \text{if} \quad J(u_0, v_0) < 0.$$

But, by condition III of section 15, we also have

$$\frac{\iint_{q_n(u_0, v_0)} J\,du\,dv}{|q_n(u_0, v_0)|} \to J(u_0, v_0).$$

Hence

$$\frac{\displaystyle\iint_{q_n(u_0,v_0)} J\,du\,dv - |S_1(q_n(u_0, v_0), T)|}{|q_n(u_0, v_0)|} \to 0 \quad \text{if} \quad J(u_0, v_0) > 0, \tag{8}$$

and

$$\frac{\displaystyle\iint_{q_n(u_0,v_0)} J\,du\,dv + |S_2(q_n(u_0, v_0), T)|}{|q_n(u_0, v_0)|} \to 0 \quad \text{if} \quad J(u_0, v_0) < 0. \tag{9}$$

By (5) in section 14 the relations (8), (9) imply that

$$\frac{\lambda(q_n(u_0, v_0))}{|q_n(u_0, v_0)|} \to 0$$

if $J(u_0, v_0) \neq 0$. If $J(u_0, v_0) = 0$, then we infer by section 13 and by condition III of section 15 that

$$0 \leqq \frac{\lambda(q_n(u_0, v_0))}{|q_n(u_0, v_0)|} \leqq \frac{\left|\displaystyle\iint_{q_n(u_0,v_0)} J\,du\,dv\right|}{|q_n(u_0, v_0)|} \to |J(u_0, v_0)| = 0.$$

17. Thus we see that for every point (u_0, v_0) of the set E of section 15 there exists a sequence of squares $q_n(u_0, v_0)$, with sides parallel to the axes and with center at (u_0, v_0), such that

$$\frac{\lambda(q_n(u_0, v_0))}{|q_n(u_0, v_0)|} \to 0 \quad \text{and} \quad |q_n(u_0, v_0)| \to 0.$$

Besides, all the squares $q_n(u_0, v_0)$ are comprised in Q. Give now $\epsilon > 0$. If we discard, for each point (u_0, v_0) of E, a finite (sufficiently large) number of the squares $q_n(u_0, v_0)$, the remaining ones will satisfy the relations

$$\lambda(q_n(u_0, v_0)) < \epsilon\,|q_n(u_0, v_0)|, \qquad |q_n(u_0, v_0)| \to 0, \qquad q_n(u_0, v_0) \subset Q.$$

The squares $q_n(u_0, v_0)$ cover the set E in the manner required by the Vitali covering theorem. We can select therefore from amongst the squares $q_n(u_0, v_0)$ a sequence of non-overlapping squares $q_1, q_2, \cdots, q_j, \cdots$ which cover E with the possible exception of a set of measure zero. Since

$$\sum q_j \subset Q \qquad \text{and} \qquad |Q - E| = 0,$$

we also have

$$|Q - \sum q_j| = 0.$$

Hence by section 13

$$\lambda(Q) \leqq \sum \lambda(q_j) \leqq \epsilon \sum |q_j| = \epsilon\,|Q| = \epsilon.$$

As ϵ was arbitrary, it follows that $\lambda(Q) = 0$. By section 13 this implies that

$$\lambda_n(Q) \to 0. \tag{10}$$

By the definition of $\lambda_n(Q)$, there exists a measurable subset V_n of Q such that

$$\left| \iint_Q J\,du\,dv - \iint_{V_n} J_n\,du\,dv \right| < \lambda_n(Q) + \frac{1}{n}.$$

(10) and (11) imply that

$$\iint_{V_n} J_n\,du\,dv \to \iint_Q J\,du\,dv,$$

and thus the theorem of section 4 is proved.

18. As far as applications are concerned, it seems that in some cases it would be sufficient to know that we have a sequence V_n such that

$$\varliminf \iint_{V_n} J_n\,du\,dv \geqq \iint_Q J\,du\,dv.$$

It is however unlikely that this relation can be derived under assumptions weaker than those needed to obtain the precise conclusion of the theorem of section 4.

THE OHIO STATE UNIVERSITY.

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ON FINSLER AND CARTAN GEOMETRIES. III

TWO-DIMENSIONAL FINSLER SPACES WITH RECTILINEAR EXTREMALS[1]

BY L. BERWALD

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Introduction. The principal part of the present paper is devoted to the problem, proposed by P. Funk,[2] of characterizing, in an invariant manner, the two-dimensional Finsler spaces the extremals of which can be given, in a suitable coordinate system, by linear equations. We call such spaces Finsler spaces with rectilinear extremals.

In an introductory section, we explain briefly, from the beginning, the theory of the two-dimensional Finsler spaces developed especially by the author[3] and by E. Cartan.[4] The standpoint of this exposition is predominantly formal. Our aim is to develop Cartan's theory of the two-dimensional Finsler spaces independently of the general theory of equivalence, and to connect it with his later theory of the n-dimensional Finsler spaces.[5] The single new feature in this section is the connection between the tensor FG^i_{jkl} and the main scalar of a two-dimensional Finsler space, given in §8.

Section II develops two different methods which lead to an invariant characterization of the two-dimensional Finsler spaces with rectilinear extremals. The first is purely analytical, and is based upon the discussion of the conditions of integrability of a certain system of partial differential equations (§§9–11). The scope of the second method is first to establish necessary and sufficient conditions in order that a two-dimensional general geometry of paths may have rectilinear paths,[6] and then to apply them to a Finsler space. This method has the advantage of showing what is the independent significance of each of the two conditions we obtain (§§12, 13).

In the third section we determine all two-dimensional Finsler spaces with rectilinear extremals the main scalar of which is a function of position only. First we establish some theorems showing that for such a space the main scalar

[1] The first two papers of this series are L. Berwald, [2], [3]. (Numbers in cornered brackets refer to the bibliography at the end of the paper).

[2] See P. Funk, [11].

[3] L. Berwald, [1].

[4] E. Cartan, [5].

[5] E. Cartan, [6].

[6] For an n-dimensional general geometry of paths ($n > 2$), J. Douglas, [7], established the corresponding conditions. When $n = 2$, only one of the two conditions for rectilinear paths results from Douglas' corresponding condition by particularization, the other does not. The reason is that, when $n = 2$, the generalized Weyl projective curvature tensor (H. Weyl, [17]; J. Douglas, [7]) vanishes identically.

is always, and the curvature nearly always, a constant (being necessarily null if the main scalar does not vanish). There exists but one exception: the case in which the main scalar has the value $\pm 3/\sqrt{2}$ (§15). This exceptional case is studied in §§16, 17. In toto, there are six types of two-dimensional Finsler spaces of the desired kind. Four of these types depend on arbitrary constants, a fifth on an arbitrary function of a single variable (§18). Finally we determine all *Landsberg spaces* with rectilinear extremals (§19).

I. Two-dimensional Finsler Spaces

1. The metric. We start with an n-dimensional manifold with coordinates x^i in which a variational problem with the fundamental integral

$$(1.1)\quad s = \int_{t_1}^{t} F(x^1, x^2, \cdots, x^n; x'^1, x'^2, \cdots, x'^n)\, dt = \int_{t_1}^{t} F(x, x')\, dt, \quad \left(x'^i = \frac{dx^i}{dt}\right)$$

is given. We suppose that F is analytic in a certain region $\mathfrak{B}$ of its $2n$ arguments to which we restrict ourselves, and that $\mathfrak{B}$ contains no points with $x'^1 = x'^2 = \cdots = x'^n = 0$. Further F is supposed to be positive and positively homogeneous of the first degree in x'. Finally, we suppose $F_1 > 0$, where

$$(1.2)\qquad F_1 = -\frac{1}{F^2}\,\text{det.}\left(\begin{array}{c|c} \dfrac{\partial^2 F}{\partial x'^i\, \partial x'^k} & \dfrac{\partial F}{\partial x'^k} \\ \hline \dfrac{\partial F}{\partial x'^i} & 0 \end{array}\right).$$

The second factor on the right of (1.2) stands for the determinant of an $(n + 1)$-rowed matrix. A manifold of the considered kind is called an *n-dimensional Finsler space*,[7] and F its *fundamental function*.

We interpret s as arc-length of the curve $x^i = x^i(t)$. As element of space we regard the oriented line-element (x, x'), that is, a point (x) and a direction $(\rho x'; \rho > 0)$ issuing from it. The quantities with which we have to deal (tensors, densities, and so on) depend exclusively on the line-element, i.e. *they are positively homogeneous of degree zero in x'*.

The metric of a Finsler space is based upon the symmetric covariant tensor of the second order

$$(1.3)\qquad g_{ik}(x, x') = \frac{1}{2}\frac{\partial^2(F^2)}{\partial x'^i\, \partial x'^k}.$$

On account of

$$(1.4)\qquad g = \text{det.}\,(g_{ik}) = F^{n+1} F_1 > 0,$$

an n-dimensional euclidean metric is associated with each line-element (x, x') by

$$(1.5)\qquad d\sigma^2 = g_{ik}(x, x')\, dx^i\, dx^k.^{8}$$

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[7] P. Finsler, [8].

[8] Repeated indices indicate summation.

This metric is particularly used for measuring the length of vectors. According to the homogeneity of F we have

$$g_{ik}x'^i x'^k = F^2, \tag{1.6}$$

$$g_{ik}x'^k = F\frac{\partial F}{\partial x'^i}. \tag{1.7}$$

Therefore orthogonality to the line-element (x, x') in the metric (1.5) is identical with transversality to the line-element.

In consequence of (1.4), the tensor g^{ik} conjugate to g_{ik} exists. By means of g_{ik} and g^{ik} the lowering and raising of indices is defined in the usual manner. We can therefore speak of covariant, contravariant, mixed components of a tensor.

The unit vector of the line-element (x, x') has the contravariant components

$$l^i = \frac{dx^i}{ds} = \frac{x'^i}{F}, \tag{1.8}$$

and the covariant components

$$l_i = \frac{\partial F}{\partial x'^i}. \tag{1.9}$$

Inner multiplication with the unit vector (l) is indicated by a zero, the position of which (above or below) does not matter; for instance

$$T_{0i} = T_{hi}l^h = T^h_{\cdot i}l_h, \qquad T^{i0} = T^{ih}l_h = T^i_{\cdot h}l^h. \tag{1.10}$$

2. Extremals and parallel displacement. The derivative Φ_s. The extremals of the variational problem $\delta s = 0$ have the differential equations

$$\frac{d}{dt}\frac{\partial F}{\partial x'^i} - \frac{\partial F}{\partial x^i} = 0. \tag{2.1}$$

With regard to $F\left(x, \frac{dx}{ds}\right) = 1$ and to the homogeneity of F, (2.1) can be written as

$$\frac{d^2x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0 \tag{2.2}$$

or

$$\frac{dl^i}{ds} + 2\frac{G^i(x, x')}{F^2(x, x')} = 0, \tag{2.3}$$

where

$$G^i(x, x') = \tfrac{1}{4}g^{ih}\left[\frac{\partial^2(F^2)}{\partial x'^h\,\partial x^m}x'^m - \frac{\partial(F^2)}{\partial x^h}\right]. \tag{2.4}$$

G^i is therefore positively homogeneous of the second degree in x'.

We indicate by Φ_s the derivative of any function $\Phi(x, x')$ with respect to the arc s of the extremal issuing from the line-element (x, x'). Because of $\frac{dF}{ds} = 0$ and of the homogeneity of G^i, we have

$$\Phi_s = \Phi_{(i)}l^i = \Phi_{(0)}, \tag{2.5}$$

where

$$\Phi_{(i)} = \frac{\partial\Phi}{\partial x^i} - \frac{\partial\Phi}{\partial x'^r}\frac{\partial G^r}{\partial x'^i}. \tag{2.6}$$

If Φ does not depend on x', $\Phi_{(i)}$ reduces to the partial derivative $\frac{\partial\Phi}{\partial x^i}$. For a scalar Φ, $\Phi_{(i)}$ are the covariant components of a vector. If Φ is a tensor which is not a scalar, the differentiation $\cdots_{(i)}$ does not generate a tensor. In particular we have

$$F_{(i)} = 0. \tag{2.7}$$

To introduce a covariant differentiation for other quantities than scalars, we define the parallel displacement of line-elements and the parallel displacement of quantities from one line-element to another parallel. By analogy with the Riemann space, it is natural to take as coefficients of a connection in the Finsler space the functions

$$\begin{cases} \Gamma^{*k}_{ij} = \Gamma^{*k}_{ji} = g^{kh}\Gamma^{*}_{ihj}, \\ \text{where} \\ \Gamma^{*}_{ihj} = \tfrac{1}{2}(g_{ih(j)} + g_{jh(i)} - g_{ij(h)}). \end{cases} \tag{2.8}$$

From (1.3) and (2.4) it follows that

$$\Gamma^{*k}_{ij}x'^i = \frac{\partial G^k}{\partial x'^j}, \qquad \Gamma^{*k}_{ij}x'^i x'^j = 2G^k. \tag{2.9}$$

The parallel displacement of the line-element (x, x') to the point $(x + dx)$ is now defined by

$$dl^i + l^h\Gamma^{*i}_{hj}\,dx^j = dl^i + \frac{1}{F}\frac{\partial G^i}{\partial x'^j}dx^j = 0, \tag{2.10}$$

and the parallel displacement of an arbitrary vector X^i from its line-element (x, x') to the line-element obtained from (x, x') by parallel displacement to the point $(x + dx)$ by

$$dX^i + X^h\Gamma^{*i}_{hj}\,dx^j = 0. \tag{2.11}$$

The left member of (2.11) is the invariant differential DX^i of the vector X^i. For any quantity, it is defined in a corresponding way, as is well known. The

coefficient of dx^j in the invariant differential of a quantity Φ is called the covariant derivative $\Phi_{|j}$ of the quantity. For instance, we have

$$(2.12)\qquad \begin{cases} X^i_{|j} = X^i_{(j)} + X^h \Gamma^{*i}_{hj}, \\ T_{ik|j} = T_{ik(j)} - T_{rk}\Gamma^{*r}_{ij} - T_{ir}\Gamma^{*r}_{kj}, \end{cases}$$

where $X^i_{(j)}$, $T_{ik(j)}$ are given by (2.6). (2.12) defines the covariant derivative also for sets of functions which behave like tensors under transformations of coordinates, but are not homogeneous of degree zero in x'. For a scalar, the differentiations $\cdots_{(i)}$ and $\cdots_{|i}$ are identical.

From the definition of the covariant derivative and from (2.7), (2.10)–(2.12), it follows immediately that

$$(2.13)\qquad \begin{cases} \text{(a)}\quad F_{|j} = 0, \qquad \text{(b)}\quad g_{ik|j} = 0, \\ \text{(c)}\quad l^i_{|j} = 0, \qquad l_{i|j} = 0. \end{cases}$$

Besides the covariant derivative we use in the following the derivative

$$(2.14)\qquad \Phi_{||j} = F\frac{\partial\Phi}{\partial x'^j}.$$

For a function Φ homogeneous of degree zero in x', we have

$$(2.15)\qquad \Phi_{||0} = 0.$$

Both differentiations $\cdots_{|j}$ and $\cdots_{||j}$ generate from a given quantity a new quantity of the same kind having a subscript more.

Finally, we state the *connection between* Γ^{*k}_{ij} *and* G^k. For that purpose we consider the symmetric tensor of the third order

$$(2.16)\qquad A_{ikj} = \tfrac{1}{2}g_{ik||j} = \tfrac{1}{4}F\frac{\partial^3(F^2)}{\partial x'^i\,\partial x'^k\,\partial x'^j}.$$

We have then

$$(2.17)\qquad A^{ik}_j = -\tfrac{1}{2}g^{ik}_{||j},$$

and with regard to the homogeneity of F

$$(2.18)\qquad A_{ik0} = A_{i0j} = A_{0kj} = 0.$$

If we write

$$(2.19)\qquad \frac{\partial^2 G^k}{\partial x'^i\,\partial x'^j} = G^k_{ij},$$

the connection sought is given by

$$(2.20)\qquad \Gamma^{*k}_{ij} = G^k_{ij} - A^k_{ij|0}.\,[9]$$

[9] See E. Cartan, [6], VIII.

3. Permutation formulas. The differentiations $\cdots_{(j)}$ and $\cdots_{||j}$ satisfy the permutation formulas

$$\Phi_{(i)(k)} - \Phi_{(k)(i)} = -\Phi_{||r} R^r_{0ik}, \tag{3.1}$$

where

$$R^r_{0ik} = \frac{1}{F}\left(\frac{\partial^2 G^r}{\partial x'^i \partial x^k} - \frac{\partial^2 G^r}{\partial x'^k \partial x^i} - G^r_{is}\frac{\partial G^s}{\partial x'^k} + G^r_{ks}\frac{\partial G^s}{\partial x'^i}\right), \tag{3.2}$$

and

$$\Phi_{(i)||k} - \Phi_{||k(i)} = -\Phi_{||r}(A^r_{ik|0} + \Gamma^{*r}_{ik}). \tag{3.3}$$

R^r_{0ik} is a tensor which results from Cartan's curvature tensor R^r_{hik} by inner multiplication with l^h.[10] (3.3) is obtained by means of (2.7) and (2.20).

4. Two-dimensional Finsler spaces. Normal vector and main scalar. In a two-dimensional Finsler space, besides the unit vector (l), the *normal vector*, i.e., the unit vector orthogonal to (l), is of importance. If h_i are its covariant, h^i its contravariant components, we have

$$\text{(a)}\ g_{ik} = l_i l_k + h_i h_k, \tag{4.1}$$
$$\text{(b)}\ \delta^k_i = l_i l^k + h_i h^k, \quad (\delta^k_i = 1 \text{ for } i = k, = 0 \text{ for } i \neq k).$$

The connection between the vectors (l) and (h) is given with the aid of the ϵ-tensor the components of which are

$$\begin{cases} \epsilon_{11} = 0, & \epsilon_{12} = \sqrt{g}, & \epsilon_{21} = -\sqrt{g}, & \epsilon_{22} = 0, \\ \epsilon^{11} = 0, & \epsilon^{12} = \dfrac{1}{\sqrt{g}}, & \epsilon^{21} = -\dfrac{1}{\sqrt{g}}, & \epsilon^{22} = 0. \end{cases} \tag{4.2}$$

Here and later on, $\sqrt{\ }$ indicates the *positive* root. It is seen that

$$\epsilon_{ik} + \epsilon_{ki} = 0, \qquad \epsilon^{ik}\epsilon_{jk} = \delta^i_j, \qquad \epsilon^{ik}\epsilon_{ik} = 2. \tag{4.3}$$

Now we obtain, if the orientation of the normal vector is suitably chosen

$$\begin{cases} \text{(a)} & l^i = \epsilon^{ik} h_k, & l_i = \epsilon_{ik} h^k, \\ \text{(b)} & h^i = -\epsilon^{ik} l_k, & h_i = -\epsilon_{ik} l^k, \\ \text{(c)} & l^i h^k - l^k h^i = \epsilon^{ik}, & l_i h_k - l_k h_i = \epsilon_{ik}. \end{cases} \tag{4.4}$$

Using (2.13, b, c), we have $\epsilon_{ik|j} = \epsilon^{ik}_{|j} = 0$ and therefore

$$h^i_{|j} = 0, \qquad h_{i|j} = 0. \tag{4.5}$$

The normal vector can be used for expressing the tensor A_{ikj} by a scalar. With regard to (2.18) we can put

$$A_{ikj} = \tfrac{1}{2} g_{ik||j} = I h_i h_k h_j. \tag{4.6}$$

[10] E. Cartan, [6], formulas (XIX), (XX), p. 36.

Hence we have in particular

$$(\sqrt{g})_{||j} = I\sqrt{g}\, h_j. \tag{4.7}$$

I is called the *main scalar* of the two-dimensional Finsler space.[11]

5. The derivative Φ_b. By means of the normal vector we define the derivative

$$\Phi_b = \Phi_{(i)}h^i. \tag{5.1}$$

Then we have conversely

$$\Phi_{(i)} = \Phi_s l_i + \Phi_b h_i. \tag{5.2}$$

For a function Φ positively homogeneous of degree zero in x', the derivative Φ_b corresponds to the difference of its values in the line-element (x, x') and in the line-element obtained by parallel displacement in transversal direction. Since from (2.13c), (4.5) it follows that

$$l^i_{(j)} = -\frac{1}{F}\frac{\partial G^i}{\partial x'^j}, \qquad h^i_{(j)} = -h^r\Gamma^{*i}_{rj} \tag{5.3}$$

and

$$l_{i(j)} = l_{j(i)} = l_r G^r_{ij}, \qquad h_{i(j)} = h_{j(i)} = h_r\Gamma^{*r}_{ij}, \tag{5.4}$$

we get

$$l^i_s = -2\frac{G^i}{F^2}, \qquad l^i_b = h^i_s = -\frac{1}{F}\frac{\partial G^i}{\partial x'^r}h^r, \qquad h^i_b = -\Gamma^{*i}_{rs}h^r h^s, \tag{5.5}$$

and

$$\begin{cases} (l_i)_s = \dfrac{1}{F}\dfrac{\partial G^r}{\partial x'^i}l_r, & (l_i)_b = G^r_{is}l_r h^s, \\[2ex] (h_i)_s = \dfrac{1}{F}\dfrac{\partial G^r}{\partial x'^i}h_r, & (h_i)_b = \Gamma^{*r}_{is}h_r h^s. \end{cases} \tag{5.6}$$

The first equation (5.5) is taken from (2.3).

6. Landsberg's angle and the derivative Φ_ϑ. In a two-dimensional Finsler space, we introduce for functions Φ, which are *positively homogeneous of degree zero in x'*, in addition to the derivatives Φ_s, Φ_b the derivative Φ_ϑ with respect to Landsberg's angle. Landsberg[12] defines the angle at the point (x) by the integral

$$\vartheta = \int \epsilon_{ik}\frac{x'^i dx'^k}{F^2} = \int h_k\, dl^k \tag{6.1}$$

[11] L. Berwald, [1], p. 204. There $\mathfrak{J} = \frac{1}{2}I$ is called the main scalar.
[12] G. Landsberg, [12].

up to an arbitrary additive function of position. The angle between two line-elements $(x, \bar{x}')$ and $(x, \bar{\bar{x}}')$ at the point (x) is defined by the definite integral (6.1), taken from $\bar{x}'^2:\bar{x}'^1$ to $\bar{\bar{x}}'^2:\bar{\bar{x}}'^1$.

Because of (6.1) we have $\vartheta_{||i} = h_i$ and therefore, for a function Φ positively homogeneous of degree zero in x'

$$\Phi_{||i} = \Phi_\vartheta h_i\,, \qquad \Phi_\vartheta = \Phi_{||i} h^i. \tag{6.2}$$

From (4.7) and (4.2) it follows in particular that

$$\begin{cases} (g_{ik})_\vartheta = 2Ih_ih_k\,, \\ (\sqrt{g})_\vartheta = I\sqrt{g}, \qquad \sqrt{g} = \varphi(x)e^{\int I\,d\vartheta}, \end{cases} \tag{6.3}$$

where $\varphi(x)$ is an arbitrary function of position, and

$$(\epsilon_{ik})_\vartheta = I\epsilon_{ik}\,, \qquad (\epsilon^{ik})_\vartheta = -I\epsilon^{ik}. \tag{6.4}$$

Further we get from (1.8), (1.9), (4.4)

$$\begin{cases} l^i_\vartheta = h^i, \qquad h^i_\vartheta = -l^i - Ih^i, \\ (l_i)_\vartheta = h_i\,, \qquad (h_i)_\vartheta = -l_i + Ih_i\,. \end{cases} \tag{6.5}$$

7. Curvature of a two-dimensional Finsler space. Cartan's permutation formulas. On account of (2.13b), the parallel displacement (2.11) is euclidean (or metrical). Therefore the curvature tensor $R_{hjik} = g_{rj}R^r_{hik}$ is skew-symmetric in the two first indices also.[13] Hence we have $R^r_{0ik}l_r = 0$ and

$$R^r_{0ik} = Kh^r\epsilon_{ik}\,. \tag{7.1}$$

The scalar

$$K = \tfrac{1}{2}R^r_{0ik}h_r\epsilon^{ik} \tag{7.2}$$

is called the *curvature* of the Finsler space.[14]

From the permutation formulas (3.1), (3.2) we can now derive *Cartan's permutation formulas*,[15] which are fundamental for the two-dimensional Finsler space. Above all we have from (4.5) and (4.6)

$$A^r_{ik|0} = I_s h_i h^r h_k\,. \tag{7.3}$$

Now let Φ be a function positively homogeneous of degree zero in x'. From (3.1), (3.3) and (7.1), (7.3) (6.2) it follows that

$$\Phi_{(i)(k)} - \Phi_{(k)(i)} = -\Phi_\vartheta K\epsilon_{ik}\,, \tag{7.4}$$

$$\Phi_{(i)||k} - \Phi_{||k(i)} = -\Phi_\vartheta(I_s h_i h_k + h_r\Gamma^{*r}_{ik}). \tag{7.5}$$

[13] E. Cartan, [6], p. 36.

[14] K was introduced by A. L. Underhill, [16].

[15] E. Cartan, [5], p. 121; see also P. Funk, [11].

By going back to the definitions of Φ_s, Φ_b, Φ_ϑ, we obtain with regard to (5.5) and (6.5) *Cartan's permutation formulas*

$$\boxed{\begin{aligned} \Phi_{b\vartheta} - \Phi_{\vartheta b} &= -\Phi_s - I\Phi_b - I_s\Phi_\vartheta, \\ \Phi_{\vartheta s} - \Phi_{s\vartheta} &= \qquad - \Phi_b \\ \Phi_{sb} - \Phi_{bs} &= \qquad\qquad - K\Phi_\vartheta. \end{aligned}} \tag{7.6}$$

From (7.6) and from Jacobi's identity

$$\{(\Phi_{sb\vartheta} - \Phi_{bs\vartheta}) - (\Phi_{\vartheta sb} - \Phi_{\vartheta bs})\} + \text{cycl.} = 0, \tag{7.7}$$

where + cycl. stands for the terms which arise by cyclical permutation of s, b, ϑ from the term written, we obtain, by cancelling a factor Φ_ϑ, the "Bianchi" identity[16]

$$\boxed{I_{ss} + IK + K_\vartheta = 0.} \tag{7.8}$$

8. The tensor FG^i_{jkr}. Affinely connected Finsler spaces. In the following we put, for the sake of simplicity,

$$\frac{\partial^3 G^i}{\partial x'^j \partial x'^k \partial x'^r} = G^i_{jkr}, \qquad \frac{\partial^4 G^i}{\partial x'^j \partial x'^k \partial x'^r \partial x'^t} = G^i_{jkrt} \tag{8.1}$$

In consequence of the homogeneity of G^i we have $G^i_{0kr} = G^i_{j0r} = G^i_{jk0} = 0$. Therefore if we decompose the tensor FG^i_{jkr} in components according to the vectors (l) and (h), there will appear only terms with $l^ih_jh_kh_r$ and $h^ih_jh_kh_r$.

By carrying through this decomposition, we find first from (2.19), (2.20), (7.3), (6.2)

$$FG^i_{jkr} = \Gamma^{*i}_{jk}\,{}_{||r} + (I_sh^ih_jh_k)_\vartheta h_r. \tag{8.2}$$

With regard to (6.5) we obtain

$$(I_sh^ih_jh_k)_\vartheta = (I_{s\vartheta} + II_s)h^ih_jh_k - I_s(l^ih_jh_k + h^il_jh_k + h^ih_jl_k). \tag{8.3}$$

On the other hand, we have from (2.8), (2.17), (4.6)

$$\Gamma^{*i}_{jk}\,{}_{||r} = g^{ih}\Gamma^*_{jhk||r} - 2Ih^ih_rh_m\Gamma^{*m}_{jk}. \tag{8.4}$$

From (2.8), (7.5) and (4.6), (5.4) we get for the first term on the right of (8.4)

$$\begin{aligned} g^{ih}\Gamma^*_{jhk||r} = [(I_b - II_s)h^ih_jh_k - I_s(l^ih_jh_k \\ - h^il_jh_k - h^ih_jl_k)]h_r + 2Ih^ih_rh_m\Gamma^{*m}_{jk}. \end{aligned} \tag{8.5}$$

Hence we obtain finally

$$\boxed{FG^i_{jkr} = [-2I_sl^i + (I_{s\vartheta} + I_b)h^i]h_jh_kh_r.} \tag{8.6}$$

---

[16] It appears first, expressed otherwise, in L. Berwald, [1], formula (78), p. 206.

In particular we have

$$FG^r_{jkr} = (I_{s\vartheta} + I_b)h_jh_k\,. \tag{8.7}$$

By operating on (8.7) with $\cdots\|_m = h_m \dfrac{\partial}{\partial\vartheta}$, it follows with regard to (1.9), (6.5) that

$$\begin{aligned} F^2G^r_{jkmr} = {} & [(I_{s\vartheta} + I_b)_\vartheta + 2I(I_{s\vartheta} + I_b)]h_jh_kh_m \\ & -(I_{s\vartheta} + I_b)(l_jh_kh_m + h_jl_kh_m + h_jh_kl_m). \end{aligned} \tag{8.8}$$

A Finsler space is said to be *affinely connected*, if G^i are quadratic polynomials in x'. From (8.6) it is seen that the two-dimensional affinely connected Finsler spaces are characterized by

$$I_s = I_b = 0. \tag{8.9}$$

According to the third permutation formula (7.6) we have for these spaces either $K = 0$ or $I =$ const. From (3.2), (7.2) we see that, for an affinely connected Finsler space with $K = 0$, the G^i_{jk}, which are functions of position only, can all be transformed together to zero by a transformation of coordinates. Then it follows from (2.7) that F does not depend on x. The converse is evident. Hence the affinely connected Finsler spaces with $K = 0$ are identical with the *Minkowski spaces.* The two-dimensional affinely connected Finsler spaces with $I =$ const. were determined by the author. In §14 we come back to these spaces.[17]

II. Necessary and Sufficient Conditions for Two-dimensional Finsler Spaces With Rectilinear Extremals

9. Differential equations for the vectors (l) and (h) in a two-dimensional Finsler space with rectilinear extremals. The definition of a Finsler space with rectilinear extremals was given in the introduction. In the coordinate system mentioned there, the differential equations of the extremals are

$$x'^i x''^k - x'^k x''^i = 0, \qquad \left(x'^m = \frac{dx^m}{dt},\, x''^m = \frac{d^2x^m}{dt^2}\right). \tag{9.1}$$

For a two-dimensional Finsler space there exists but one such equation. In this section, we seek conditions, invariant under transformation of coordinates, which are necessary and sufficient in order that a two-dimensional Finsler space may have rectilinear extremals.

First we suppose that the considered Finsler space has rectilinear extremals. Let x^i be coordinates for which the extremals are given by (9.1). It is evident that (9.1) is equivalent to

$$G^i = F^2ql^i, \tag{9.2}$$

[17] Cf. L. Berwald, [1], pp. 207 f., 212 f., 215 ff.; E. Cartan, [5], p. 134 f.

where q is a function of x, x', positively homogeneous of degree zero in x'. The The basis of our considerations is the system of equations (5.5), (6.5). On account of (9.2), it can be written as

$$l^i_s = -2ql^i, \qquad l^i_b = -q_\vartheta l^i - qh^i, \qquad l^i_\vartheta = h^i; \tag{9.3}$$

$$\begin{gathered} h^i_s = -q_\vartheta l^i - qh^i, \qquad h^i_b = -(q + Iq_\vartheta + q_{\vartheta\vartheta})l^i - (2q_\vartheta - I_s)h^i, \\ h^i_\vartheta = -l^i - Ih^i. \end{gathered} \tag{9.4}$$

In order to derive (9.3), (9.4), we can proceed from the first and the last equation (9.3). Then the second permutation formula (7.6) gives the second equation (9.3), which is identical with the first equation (9.4). By the same procedure we obtain the second equation (9.4) from the first and the third.

Let us now consider the conditions of integrability of (9.3), (9.4). Two of them are identically satisfied, in consequence of the derivation of (9.3), (9.4) just given. The condition of integrability, which follows from the permutation formula for $l^i_{b\vartheta} - l^i_{\vartheta b}$, is also identically satisfied. Each of the other permutation formulas gives two conditions of integrability, since the coefficients of l^i and h^i must be respectively equal in both members. Thus we obtain six conditions of integrability:

$$q_{\vartheta s} = 2q_b + qq_\vartheta, \tag{9.5}$$

$$q^2 + q_s = -K, \tag{9.6}$$

$$q_{\vartheta b} = (q + Iq_\vartheta + q_{\vartheta\vartheta})_s - q(q + Iq_\vartheta + q_{\vartheta\vartheta}) + I_s q_\vartheta - K - q_\vartheta^2, \tag{9.7}$$

$$qq_\vartheta - q_b + 2q_{\vartheta s} = I_{ss} + KI = -K_\vartheta, \tag{9.8}$$

$$(q + Iq_\vartheta + q_{\vartheta\vartheta})_\vartheta + 2I(q + Iq_\vartheta + q_{\vartheta\vartheta}) + 2I_s = 0, \tag{9.9}$$

$$3(q + Iq_\vartheta + q_{\vartheta\vartheta}) = I_{s\vartheta} + I_b. \tag{9.10}$$

In (9.8) we made use of the identity (7.8).

10. Reduction of the integrability conditions. In the first place, we show that the conditions of integrability (9.5)–(9.8) can be substituted by the two equations

$$K = -q^2 - q_s, \tag{10.1}$$

$$K_\vartheta = -3(q_b + qq_\vartheta), \tag{10.2}$$

the second of which is found by substituting the value of $q_{\vartheta s}$ from (9.5) in (9.8).

Indeed, (9.5), (9.7), (9.8) follow from (10.1), (10.2). For, by substituting the value of K_ϑ from (10.1) in (10.2) and making use of the second formula (7.6), we obtain (9.5), and by substituting the value of $3(q_b + qq_\vartheta)$ from (9.5) in (10.2), we get (9.8). In order to derive (9.7) we differentiate (9.5) with respect to ϑ and apply the second permutation formula (7.6) for $\Phi = q_\vartheta$. Thus we find

$$q_{\vartheta\vartheta s} = 2q_{b\vartheta} - q_{\vartheta b} + q_\vartheta^2 + qq_{\vartheta\vartheta} = 2(q_{b\vartheta} - q_{\vartheta b}) + q_{\vartheta b} + q_\vartheta^2 + qq_{\vartheta\vartheta}.$$

Now apply the first permutation formula (7.6) to $q_{b\vartheta} - q_{\vartheta b}$ and form $(q + Iq_\vartheta + q_{\vartheta\vartheta})_s$. Then equation (9.7) results easily, in consequence of (9.5) and (10.1).

11. Necessary and sufficient conditions for rectilinear extremals. From §10 we see that the conditions of integrability (9.5)–(9.10) of the system (9.3), (9.4) of partial differential equations may be reduced to (10.1), (10.2), (9.10), (9.9). The first three of these equations can be written as

$$q_s = -K - q^2, \tag{11.1}$$

$$q_b = -\tfrac{1}{3}K_\vartheta - qq_\vartheta, \tag{11.2}$$

$$q_{\vartheta\vartheta} = \tfrac{1}{3}(I_{s\vartheta} + I_b) - Iq_\vartheta - q. \tag{11.3}$$

The equations (9.10), (9.9) are equivalent to (11.3) and

$$(I_{s\vartheta} + I_b)_\vartheta + 2I(I_{s\vartheta} + I_b) + 6I_s = 0. \tag{11.4}$$

(11.1)–(11.3) are a system of partial differential equations for the function q. In consequence of the permutation formulas (7.6), the conditions of integrability of this system are obtained by calculating the expressions

$$A = q_{sb} - q_{bs} + Kq_\vartheta,$$

$$B = q_{\vartheta s\vartheta} - q_{\vartheta\vartheta s} - q_{\vartheta b},$$

$$C = q_{\vartheta b\vartheta} - q_{\vartheta\vartheta b} + q_{\vartheta s} + Iq_{\vartheta b} + I_s q_{\vartheta\vartheta}$$

with the aid of (11.1)–(11.3) and by equating the results to zero. The calculation of A, B, C is simplified by the remark that $q_{\vartheta s}$ is given by (9.5), $q_{\vartheta b}$ by (9.7), if q_b, $q + Iq_\vartheta + q_{\vartheta\vartheta}$ are respectively replaced by their values (11.2), (9.10).

A simple calculation shows that $A = 0$ reduces to

$$K_{\vartheta s} - 3K_b = 0. \tag{11.5}$$

For B, we find first

$$B = -\tfrac{2}{3}[K_{\vartheta\vartheta} + (I_{s\vartheta} + I_b)_s + IK_\vartheta].$$

Since the permutation formulas (7.6) give

$$I_{s\vartheta s} + I_{bs} = I_{ss\vartheta} - I_{sb} + I_{bs} = I_{ss\vartheta} + KI_\vartheta,$$

we have further

$$B = -\tfrac{2}{3}(K_\vartheta + IK + I_{ss})_\vartheta.$$

Because of the Bianchi identity (7.8), B is therefore identically zero. For the calculation of C, the second permutation formula (7.6) gives

$$(I_{s\vartheta} + I_b)_{s\vartheta} - (I_{s\vartheta} + I_b)_b = (I_{s\vartheta} + I_b)_{\vartheta s}.$$

By means of this formula, we obtain

$$C = \tfrac{1}{3}[(I_{s\vartheta} + I_b)_\vartheta + 2I(I_{s\vartheta} + I_b)]_s - \tfrac{1}{3}q[(I_{s\vartheta} + I_b)_\vartheta + 2I(I_{s\vartheta} + I_b) + 6I_s] - 2K_\vartheta - 2IK,$$

or, with regard to (7.8)

$$C = \tfrac{1}{3}[(I_{s\vartheta} + I_b)_\vartheta + 2I(I_{s\vartheta} + I_b) + 6I_s]_s - \tfrac{1}{3}q[(I_{s\vartheta} + I_b)_\vartheta + 2I(I_{s\vartheta} + I_b) + 6I_s].$$

Consequently C is zero, if (11.4) holds. Hence we have:

If (11.4), (11.5) *are satisfied, the system* (11.1)–(11.3) *is completely integrable.*

It is evident that the equations (11.4), (11.5) are necessary in order that a two-dimensional Finsler space may have rectilinear extremals. They are also sufficient. For let (11.4), (11.5) hold. Then the system (11.1)–(11.3) is completely integrable. For a solution q of this system, all conditions of integrability (9.5)–(9.10) of the system (9.3), (9.4) are satisfied. But the first equation (9.3) or

$$x^i_{ss} + 2qx^i_s = 0 \tag{11.6}$$

states that the considered two-dimensional Finsler space has rectilinear extremals. Therefore we have:

A necessary and sufficient condition in order that the extremals of a two-dimensional Finsler space be rectilinear is that the equations

$$\boxed{(I_{s\vartheta} + I_b)_\vartheta + 2I(I_{s\vartheta} + I_b) + 6I_s = 0} \tag{11.4}$$

and

$$\boxed{K_{\vartheta s} - 3K_b = 0} \tag{11.5}$$

be satisfied.[18]

12. Geometrical meaning of (11.4). In the foregoing paragraphs, we established the conditions (11.4), (11.5) in a purely analytical way which does not show the independent geometrical meaning of each of these conditions. We give therefore still another deduction, which does not possess this disadvantage.

For that purpose we insert some general considerations, restricting ourselves to the case of two dimensions, for the sake of brevity. Let a two-dimensional manifold with coordinates x^i be given and a differential equation

$$x'^h(x''^i + 2G^i(x, x')) - x'^i(x''^h + 2G^h(x, x')) = 0, \qquad \left(x'^m = \frac{dx^m}{dt},\ x''^m = \frac{d^2x^m}{dt^2}\right), \tag{12.1}$$

[18] Cf. P. Funk, [11].

where the functions $G^i(x, x')$ are analytic and positively homogeneous of the second degree in x'.[19] (12.1) defines a system of general *paths* $x^i = x^i(t)$ in the manifold. This system remains unaltered under the transformation

$$\bar{G}^i(x, x') = G^i(x, x') + p(x, x')x'^i, \tag{12.2}$$

where p is analytic and positively homogeneous of the first degree in x'.[19] We call (12.2) a *projective change* of the functions G^i. A transformation of coordinates $\overset{*}{x}{}^i = \overset{*}{x}{}^i(x)$ transforms $2G^i$ in

$$2\overset{*}{G}{}^h = \left(2G^i + \frac{\partial^2 x^i}{\partial \overset{*}{x}{}^j \partial \overset{*}{x}{}^k} \overset{*}{x}{}'^j \overset{*}{x}{}'^k\right) \frac{\partial \overset{*}{x}{}^h}{\partial x^i}. \tag{12.3}$$

The second and the higher derivatives

$$\frac{\partial^2 G^i}{\partial x'^j \partial x'^k} = G^i_{jk}; \qquad \frac{\partial^3 G^i}{\partial x'^j \partial x'^k \partial x'^s} = G^i_{jks}, \qquad \frac{\partial^4 G^i}{\partial x'^j \partial x'^k \partial x'^s \partial x'^m} = G^i_{jksm} \tag{12.4}$$

therefore transform respectively like the coefficients of an affine connection (depending upon the line-element), and like tensors. To the projective change (12.2) of G^i corresponds respectively the projective change

$$\text{(12.5)} \begin{cases} \text{(a)} \quad \bar{G}^i_{jk} = G^i_{jk} + \dfrac{\partial p}{\partial x'^j} \delta^i_k + \dfrac{\partial p}{\partial x'^k} \delta^i_j + \dfrac{\partial^2 p}{\partial x'^j \partial x'^k} x'^i, \\[2ex] \text{(b)} \quad \bar{G}^i_{jks} = G^i_{jks} + \dfrac{\partial^2 p}{\partial x'^j \partial x'^k} \delta^i_s + \dfrac{\partial^2 p}{\partial x'^k \partial x'^s} \delta^i_j \\[2ex] \qquad\qquad + \dfrac{\partial^2 p}{\partial x'^s \partial x'^j} \delta^i_k + \dfrac{\partial^3 p}{\partial x'^j \partial x'^k \partial x'^s} x'^i \end{cases}$$

of the affine connection G^i_{jk}, and of the tensor G^i_{jks}.

It is evident that the extremals of a Finsler space form a system of paths, G^i being defined by (2.4) in this case. Now the meaning of the condition (11.4) can be expressed as follows:

A necessary and sufficient condition in order that in a two-dimensional Finsler space the functions G^i, defined by (2.4), *can be projectively changed in such a way that the transformed functions $\bar{G}^i$ be quadratic polynomials in x', is that equation* (11.4) holds.

PROOF: In the first place, we consider a two-dimensional space with a system of paths (12.1). Let $\bar{G}^i$ be quadratic polynomials in x. Then the right member of (12.5b) is zero. Now we have from (12.2), in consequence of the homogeneity of p

$$p = \frac{1}{3}\left(\frac{\partial \bar{G}^r}{\partial x'^r} - \frac{\partial G^r}{\partial x'^r}\right). \tag{12.6}$$

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[19] We suppose here positive homogeneity with regard to the following applications to Finsler spaces. In general, mere homogeneity is supposed.

By substituting this value in (12.5b), we find as a necessary condition for $\bar{G}^i = \bar{G}^i_{kj}(x)x'^k x'^j$

$$G^i_{jks} - \tfrac{1}{3}(G^r_{r\,jk}\delta^i_s + G^r_{rks}\delta^i_j + G^r_{rs\,j}\delta^i_k) - \tfrac{1}{3}G^r_{r\,jks}x'^i = 0. \tag{12.7}$$

Conversely, if (12.7) holds, we put

$$p = \frac{1}{3}\left(2\bar{G}^r_{kr}(x)x'^k - \frac{\partial G^r}{\partial x'^r}\right), \tag{12.8}$$

where $\bar{G}^i_{kj} = \bar{G}^i_{jk}$ are arbitrary functions of x alone. Then from (12.7) and (12.5b) we have $\bar{G}^i_{jks} = 0$. Therefore (12.7) is sufficient too for $\bar{G}^i = \bar{G}^i_{kj}(x)x'^k x'^j$.[20]

Now consider in particular a two-dimensional Finsler space, and let G^i be given by (2.4). Then, by multiplying (12.7) by $-3F$ and taking account of (8.6)–(8.8), (4.1b), we obtain (11.4) multiplied by $l^i h_j h_k h_s$.

We can give the theorem just proved the form:

A necessary and sufficient condition in order that the extremals of a two-dimensional Finsler space form a quasigeodesic system of curves is that (11.4) *be satisfied.*[21]

It is easy to prove this form of the theorem directly. We have to suppose that for the extremals $x'^i x''^k - x'^k x''^i$ is a homogeneous cubic polynomial in x', or that

$$x'^i G^k - x'^k G^i = \tfrac{1}{6}\Phi^{ik}_{jhr}(x)x'^j x'^h x'^r. \tag{12.9}$$

If we differentiate (12.9) with respect to x'^j, x'^p, x'^r, x'^s, we obtain an equation which reduces to (11.4), multiplied by $\dfrac{1}{\sqrt{g}}\, h_j h_p h_r h_s$, as can be seen from §8 and §4.

13. Geometrical meaning of (11.5). Before considering (11.5), we observe that the system of functions

$$K^i_{jk} = \frac{\partial^2 G^i}{\partial x'^j \partial x^k} - \frac{\partial^2 G^i}{\partial x'^k \partial x^j} - G^i_{jr}\frac{\partial G^r}{\partial x'^k} + G^i_{kr}\frac{\partial G^r}{\partial x'^j}, \tag{13.1}$$

where G^i are the functions which enter in (12.1), behaves by a transformation of coordinates like a tensor, but is positively homogeneous of the *first* degree in x'. We call K^i_{jk} the *fundamental curvature tensor* of the connection G^i_{jk}. For a two-dimensional Finsler space, where G^i are defined by (2.4), the fundamental curvature tensor is connected with the curvature K by

$$K^i_{jk} = FKh^i\epsilon_{jk}, \tag{13.2}$$

as (3.2) and (7.1) show.

[20] See J. Douglas, [7], p. 157 f., for $n \geqq 2$ dimensions.

[21] Cf. P. Funk, [11]. For the name "quasigeodesic system of curves" see W. Blaschke and G. Bol, [4], p. 245.

Now the geometrical meaning of (11.5) is expressed by the theorem: *In order that there may exist a projective change of the G's defined by* (2.4) *which transforms the fundamental tensor of curvature of a two-dimensional Finsler space to zero, it is necessary and sufficient that equation* (11.5) *be satisfied.*

To prove this theorem, we first establish a necessary and sufficient condition that in a two-dimensional manifold, bearing a system (12.1) of paths, the fundamental curvature tensor may be transformed to zero, by a projective change of G^i. Then we suppose that the considered manifold is a two-dimensional Finsler space, and that (12.1) is the system of its extremals (2.2), and we show that the necessary and sufficient condition we found reduces to (11.5) if we introduce the curvature K and its derivatives with respect to s, b, ϑ.

Let (12.1) be the system of paths of a two-dimensional manifold. We denote the covariant derivative with respect to the G^i_{jk} by a semicolon; for instance:

$$T_{j;k} = \frac{\partial T_j}{\partial x^k} - \frac{\partial T_j}{\partial x'^r}\frac{\partial G^r}{\partial x'^k} - T_r G^r_{jk}\,. \tag{13.3}$$

Let $\bar{K}^i_{jk}$ be the fundamental curvature tensor of the connection $\bar{G}^i_{jk}$ obtained from the connection G^i_{jk} by the projective change (12.2). Then we have

$$\begin{aligned}\bar{K}^i_{jk} = K^i_{jk} &- \left(p_{;j} - \frac{1}{2}\frac{\partial(p^2)}{\partial x'^j}\right)\delta^i_k \\ &+ \left(p_{;k} - \frac{1}{2}\frac{\partial(p^2)}{\partial x'^k}\right)\delta^i_j - \left(\frac{\partial p_{;j}}{\partial x'^k} - \frac{\partial p_{;k}}{\partial x'^j}\right)x'^i.\end{aligned} \tag{13.4}$$

Hence the postulate $\bar{K}^i_{jk} = 0$ gives

$$K^i_{jk} = \left(p_{;j} - \frac{1}{2}\frac{\partial(p^2)}{\partial x'^j}\right)\delta^i_k - \left(p_{;k} - \frac{1}{2}\frac{\partial(p^2)}{\partial x'^k}\right)\delta^i_j + \left(\frac{\partial p_{;j}}{\partial x'^k} - \frac{\partial p_{;k}}{\partial x'^j}\right)x'^i. \tag{13.5}$$

Suppose now that (13.5) is satisfied. If we put

$$K^r_{jr} = K_j\,, \qquad \frac{\partial K_r}{\partial x'^j} = K_{jr}\,, \tag{13.6}$$

we obtain by differentiation and contraction the system of differential equations

$$p_{;j} - \frac{1}{2}\frac{\partial(p^2)}{\partial x'^j} = \tfrac{2}{3}K_j + \tfrac{1}{3}K_{jr}x'^r. \tag{13.7}$$

for the function p. The condition of integrability of (13.7) is

$$\begin{aligned}p_{;j;k} - p_{;k;j} - \frac{1}{2}\left(\frac{\partial(p^2)}{\partial x'^j}\right)_{;k} + \frac{1}{2}\left(\frac{\partial(p^2)}{\partial x'^k}\right)_{;j} \\ = \tfrac{2}{3}(K_{j;k} - K_{k;j}) + \tfrac{1}{3}[(K_{jr}x'^r)_{;k} - (K_{kr}x'^r)_{;j}].\end{aligned} \tag{13.8}$$

In consequence of the permutation formulas

$$
(13.9)\qquad \begin{cases} \Phi_{;j;k} - \Phi_{;k;j} = -K^i_{jk}\,\dfrac{\partial\Phi}{\partial x'^i}, \\[2ex] \left(\dfrac{\partial\Phi}{\partial x'^j}\right)_{;k} - \dfrac{\partial\Phi_{;k}}{\partial x'^j} = 0, \end{cases}
$$

and of the homogeneity of p, we find for the left member of (13.8) the value

$$
-\frac{\partial p}{\partial x'^i}\left\{K^i_{jk} - \left(p_{;j} - \frac{1}{2}\frac{\partial(p^2)}{\partial x'^j}\right)\delta^i_k + \left(p_{;k} - \frac{1}{2}\frac{\partial(p^2)}{\partial x'^k}\right)\delta^i_j - \left(\frac{\partial p_{;j}}{\partial x'^k} - \frac{\partial p_{;k}}{\partial x'^j}\right)x'^i\right\},
$$

which is zero, because of (13.5). Hence the condition of integrability of (13.7) becomes finally

$$
(13.10)\qquad \tfrac{2}{3}(K_{j;k} - K_{k;j}) + \tfrac{1}{3}[(K_{jr}x'^r)_{;k} - (K_{kr}x'^r)_{;j}] = 0.
$$

Conversely, let (13.10) be satisfied. Then the condition of integrability of (13.7) leads back to (13.5). Hence (13.10) is a necessary and sufficient condition in order that the fundamental curvature tensor of a two-dimensional manifold with a system (12.1) of paths can be transformed to zero by a projective change of G^i.[22]

If the curves (12.1) are the extremals of a two-dimensional Finsler space, we have with regard to (2.6), (13.3) and to the symmetry $G^i_{jk} = G^i_{kj}$

$$
(13.11)\qquad T_{j;k} - T_{k;j} = T_{j(k)} - T_{k(j)}.
$$

Further it follows from (13.2), (4.4), (6.2), (6.5), (4.1) that

$$
(13.12)\qquad K_j = FKl_j\,, \qquad K_{jr} = Kg_{rj} + K_\vartheta l_r h_j\,, \qquad K_{jr}x'^r = F(Kl_j + K_\vartheta h_j).
$$

If we introduce, by means of (5.2), the derivatives with respect to s and b instead of the derivative $\cdots_{(i)}$, we obtain in consequence of (2.7), (5.4), (4.4) and of the symmetry $G^i_{jk} = G^i_{kj}$

$$
(13.13)\qquad \begin{cases} K_{j(k)} - K_{k(j)} = FK_b(l_j h_k - l_k h_j) = FK_b\,\epsilon_{jk}\,, \\ (K_{rj}x'^r)_{(k)} - (K_{rk}x'^r)_{(j)} = F(K_b - K_{\vartheta s})\,\epsilon_{jk}\,. \end{cases}
$$

By substituting these values in (13.10) and by dropping the factor $-\frac{1}{3}F\epsilon_{jk}$, we get (11.5).

From the theorems we proved in §§12, 13, we can also easily see that the conditions (11.4) and (11.5) are necessary and sufficient in order that the

[22] In consequence of these considerations and the corresponding ones in §12, we have incidentally the theorem:

The conditions (12.7), (13.10) *are necessary and sufficient in order that a given system of paths* (12.1) *of a two-dimensional manifold be equivalent by point transformation to the straight lines of a flat projective space.*

Cf. also the end of §13. For $n > 2$ dimensions, J. Douglas, [7], p. 162 established the corresponding theorem, which is not however applicable when $n = 2$.

extremals of a two-dimensional Finsler space be rectilinear. For let the extremals be rectilinear. Then, in a suitable coordinate system, (9.2) holds. Hence we obtain from (12.2) $\bar{G}^i = 0$, by putting $p = -qF$. Therefore (12.7) and (13.10) or, what amounts to the same thing, (11.4) and (11.5) are satisfied.

Conversely, if (11.4) and (11.5) are satisfied, the coefficients $\bar{G}^i_{jk}$ of the projectively transformed affine connection (2.19) are functions of position only. It is easily seen that the curvature tensor of the connection $\bar{G}^i_{jk}$ is $\dfrac{\partial \bar{K}^i_{jk}}{\partial x'^r}$. Since (11.5) holds, this tensor is zero. Therefore the $\bar{G}^i_{jk}$ can altogether be transformed to zero by a suitable transformation of coordinates $x^i = x^i(\overset{*}{x})$. Then

$$\overset{*}{G}{}^i_{jk} = 0, \qquad 2\overset{*}{G}{}^i = \overset{*}{G}{}^i_{jk} x'^j x'^k = 0. \tag{13.14}$$

Hence, in the new coordinate system, the differential equation of the extremals has the form

$$\overset{*}{x}{}'^i \overset{*}{x}{}''^k - \overset{*}{x}{}'^k \overset{*}{x}{}''^i = 0, \tag{13.15}$$

that is, the extremals are rectilinear.

III. The Two-dimensional Finsler Spaces with Rectilinear Extremals the Main Scalar of Which is a Function of Position Only

14. The two-dimensional Finsler spaces with constant main scalar. We shall now determine all two-dimensional Finsler spaces with rectilinear extremals the main scalar of which is a function of position only. In order to avoid interruptions later on, we first reproduce briefly the manner in which *all* Finsler spaces with I = const. can be determined,[23] and we add a simple remark on these spaces.

The Finsler spaces with I = const. belong to the class of Finsler spaces for which $\dfrac{F^2}{\sqrt{g}}$ is a quadratic polynomial in x'. In fact, if we differentiate $\dfrac{F^2}{\sqrt{g}}$ three times with respect to x' and if we use (1.9), (2.14), (6.2)–6.5), we get successively

$$\frac{\partial}{\partial x'^i} \frac{F^2}{\sqrt{g}} = \frac{F}{\sqrt{g}} (2l_i - Ih_i), \tag{14.1}$$

$$\frac{\partial^2}{\partial x'^i \partial x'^k} \left(\frac{F^2}{\sqrt{g}}\right) = \frac{1}{\sqrt{g}} [2l_i l_k - I(l_i h_k + h_i l_k) + (2 - I_\vartheta) h_i h_k], \tag{14.2}$$

$$\frac{\partial^3}{\partial x'^i \partial x'^k \partial x'^j} \left(\frac{F^2}{\sqrt{g}}\right) = -\frac{1}{F\sqrt{g}} (I_{\vartheta\vartheta} + II_\vartheta) h_i h_k h_j. \tag{14.3}$$

The two-dimensional Finsler spaces, for which

$$\frac{F^2}{\sqrt{g}} = \mathfrak{g}_{ik}(x) x'^i x'^k, \qquad (\mathfrak{g}_{ik} = \mathfrak{g}_{ki}), \tag{14.4}$$

[23] L. Berwald, [1], p. 215 ff.

are therefore characterized by $I_{\vartheta\vartheta} + II_{\vartheta} = 0$. With regard to (14.2) and (4.1a), we have from (14.4)

$$\mathfrak{g}_{ik}(x) = \frac{1}{\sqrt{g}}\{g_{ik} - \tfrac{1}{2}I(l_i h_k + l_k h_i) - \tfrac{1}{2}i_{\vartheta} h_i h_k\}, \tag{14.5}$$

and from (14.5)

$$\mathfrak{g} = 1 - \tfrac{1}{4}I^2 - \tfrac{1}{2}I_{\vartheta}, \tag{14.6}$$

where $\mathfrak{g} = \det.\ (\mathfrak{g}_{ik})$.

Suppose now in particular $I = \text{const.}$ Then we have from (6.3)

$$\sqrt{g} = e^{I\vartheta} \tag{14.7}$$

since we can choose in (6.3) $\varphi(x) = 1$ without loss of generality for the following considerations. Therefore it follows from (14.4) that

$$F = \sqrt{\mathfrak{g}_{ik} x'^i x'^k}\, e^{\frac{1}{2}I\vartheta}. \tag{14.8}$$

Now it remains to determine ϑ as function of x'. For that purpose we distinguish the three cases

$$\mathfrak{g} = 1 - \tfrac{1}{4}I^2 \gtreqless 0, \tag{14.9}$$

and understand by $\alpha_i x'^i$, $\beta_i x'^i$ two linearly independent Pfaffians so that $\Delta = \alpha_1\beta_2 - \alpha_2\beta_1 > 0$. We put in the case

(a) $I^2 < 4$:

$$\mathfrak{g}_{ik} x'^i x'^k = \frac{\sqrt{\mathfrak{g}}}{\Delta}[(\alpha_i x'^i)^2 + (\beta_i x'^i)^2]; \tag{14.10a}$$

(b) $I^2 = 4$:

$$\mathfrak{g}_{ik} x'^i x'^k = \frac{1}{\Delta}(\alpha_i x'^i)^2; \tag{14.10b}$$

(c) $I^2 > 4$:

$$\mathfrak{g}_{ik} x'^i x'^k = \frac{2\sqrt{-\mathfrak{g}}}{\Delta}(\alpha_i x'^i)(\beta_k x'^k). \tag{14.10c}$$

Because of (6.1) and (14.1) we have

$$\vartheta = \int \frac{x'^1 dx'^2 - x'^2 dx'^1}{\mathfrak{g}_{ik}(x) x'^i x'^k}. \tag{14.11}$$

By integrating, we get respectively in the cases (a), (b), (c):

$$\vartheta = \frac{1}{\sqrt{\mathfrak{g}}}\operatorname{arctg}\frac{\beta_i x'^i}{\alpha_i x'^i} + \psi(x), \tag{14.12a}$$

$$\vartheta = \frac{\beta_i x'^i}{\alpha_i x'^i} + \psi(x), \tag{14.12b}$$

$$\vartheta = \frac{1}{2\sqrt{-\mathfrak{g}}} \log \frac{\beta_i x'^i}{\alpha_i x'^i} + \psi(x), \tag{14.12c}$$

where $\psi(x)$ is an arbitrary function of position. By substituting these values in (14.8), we obtain finally, by means of (14.9)–(14.12), the following types of *two-dimensional Finsler spaces with I = const.*

$$I^2 < 4 \quad : \quad F = ((\alpha_i x'^i)^2 + (\beta_i x'^i)^2)^{\frac{1}{2}} e^{\frac{I}{\sqrt{(4-I^2)}} \operatorname{arctg} \frac{\beta_i x'^i}{\alpha_i x'^i}}, \tag{14.13a}$$

$$I^2 = 4 \quad : \quad F = (\alpha_i x'^i) e^{\frac{1}{2} I \frac{\beta_i x'^i}{\alpha_i x'^i}}, \tag{14.13b}$$

$$I^2 > 4 \quad : \quad F = (\alpha_i x'^i)^{\frac{1}{2}\left(1 - \frac{I}{\sqrt{(I^2-4)}}\right)} (\beta_i x'^i)^{\frac{1}{2}\left(1 + \frac{I}{\sqrt{(I^2-4)}}\right)}. \tag{14.13c}$$

In consequence of (14.10) and (14.12) a multiplicative positive function of position would enter in (14.13). We have taken it in in α_i, β_i.

For the Finsler spaces with I = const. we have the following theorem:

If the curvature of a two-dimensional Finsler space with constant $I \neq 0$ is a function of position only, the space is a Minkowski space.

Indeed, because of I = const. the Finsler space is affinely connected (§8). With regard to I = const., $I \neq 0$, $K_\vartheta = 0$, the Bianchi identity (7.8) gives $K = 0$. Hence the space is a Minkowski space (§8).

15. Theorems on two-dimensional Finsler space with rectilinear extremals. In the first place, we establish some theorems which hold for Finsler spaces with rectilinear extremals.

THEOREM 1. *When for a two-dimensional Finsler space with a quasigeodesic system of extremals the main scalar is a function of position only, it is constant.*

PROOF: From the hypothesis

$$I_\vartheta = 0 \tag{15.1}$$

it follows, with the aid of the first two permutation formulas (7.6), that

$$I_{s\vartheta} = I_b, \qquad I_{b\vartheta} = -I_s - II_b. \tag{15.2}$$

By substituting these values in (11.4), we have

$$II_b + 2I_s = 0. \tag{15.3}$$

If we differentiate (15.3) with respect to ϑ and take note of (15.1), (15.2), we obtain

$$(I^2 - 2)I_b + II_s = 0. \tag{15.4}$$

The determinant of the equations (15.3), (15.4) for I_b, I_s is $4 - I^2$. Therefore we have either $I^2 = 4$ or $I_b = I_s = 0$, that is, in each case, I = const.

THEOREM 2. *A two-dimensional Finsler space with rectilinear extremals for which I^2 is a function of position only and $\neq \frac{9}{2}$, has constant curvature.*

Proof: With regard to theorem 1 we have by hypothesis

$$(15.5)\qquad (a)\ I = \text{const.}, \qquad (b)\ K_{\vartheta s} = 3K_b .$$

Because of (15.5a) the Bianchi identity (7.8) gives

$$(15.6)\qquad K_\vartheta = -IK.$$

In consequence of I = const., it follows from (15.6) that

$$(15.7)\qquad (a)\ K_{\vartheta s} = -IK_s, \qquad (b)\ K_{\vartheta b} = -IK_b .$$

By substituting the value of $K_{\vartheta s}$ from (15.5b) in (15.7a), we find

$$(15.8)\qquad IK_s + 3K_b = 0.$$

If we differentiate (15.8) with respect to ϑ, and make use of (15.5), (15.7) and of the first two permutation formulas (7.6), we obtain

$$(15.9)\qquad 3K_s + 2IK_b = 0.$$

The determinant of the equations (15.8), (15.9) for K_b, K_s is $2I^2 - 9$. Accordingly we get from (15.8), (15.9) $K_b = K_s = 0$, when $I^2 \neq \frac{9}{2}$. Finally, the last permutation formula (7.6) gives $KK_\vartheta = 0$. Hence $K = \text{const.}$

As a corollary of theorem 2 and the theorem of §14, we have

Theorem 3. *A two-dimensional Finsler space with rectilinear extremals for which I^2 is a function of position only and $\neq 0, \frac{9}{2}$ is a Minkowski space.*

For such a space is $K = 0$.

16. The exceptional case $I^2 = \frac{9}{2}$. We shall now determine the two-dimensional Finsler spaces with rectilinear extremals for which $I^2 = \frac{9}{2}$. Let us begin with some general remarks.

We write in the following

$$(16.1)\qquad x^1 = x, \qquad x^2 = y,$$

and we denote partial differentiation by subscripts. It is known that the differential equation of the extremals of a two-dimensional Finsler space can be written as follows

$$(16.2)\qquad F_{xy'} - F_{yx'} + F_1(x'y'' - y'x'') = 0,$$

where

$$(16.3)\qquad F_1 = \frac{F_{x'x'}}{(y')^2} = -\frac{F_{x'y'}}{x'y'} = \frac{F_{y'y'}}{(x')^2}.$$

From (16.3) we have

$$(16.4)\qquad F_{xy'} - F_{yx'} = 0$$

as a necessary and sufficient condition for rectilinear extremals. By differentiating (16.4) with respect to x' or y', we obtain, in consequence of (16.3), the

necessary condition

$$\frac{\partial F_1}{\partial x} x' + \frac{\partial F_1}{\partial y} y' = 0. \tag{16.5}$$

Now let us consider a two-dimensional Finsler space with $I^2 = \frac{9}{2}$. With regard to (14.13c) and (16.1), we have for such a space

$$F = \frac{(\alpha x' + \beta y')^2}{\gamma x' + \delta y'}, \qquad (\alpha\delta - \beta\gamma \neq 0). \tag{16.6}$$

We multiply the numerator and the denominator by $(\alpha\delta - \beta\gamma)^{-\frac{1}{2}}$ and put

$$\begin{aligned} A &= \frac{\alpha}{(\alpha\delta - \beta\gamma)^{\frac{1}{2}}}, & B &= \frac{\beta}{(\alpha\delta - \beta\gamma)^{\frac{1}{2}}}, \\ C &= \frac{\gamma}{(\alpha\delta - \beta\gamma)^{\frac{1}{2}}}, & D &= \frac{\delta}{(\alpha\delta - \beta\gamma)^{\frac{1}{2}}}. \end{aligned} \tag{16.7}$$

Then we obtain

$$F = \frac{(Ax' + By')^2}{Cx' + Dy'}, \tag{16.8}$$

where

$$AD - BC = 1. \tag{16.9}$$

Without loss of generality, we can suppose $D \neq 0$. For, if $D = 0$, $C \neq 0$, we arrive at $D \neq 0$ by permuting x and y. Then we have

$$A = \frac{1 + BC}{D} \tag{16.10}$$

and

$$F = \frac{1}{D^2}\left\{\frac{(x')^2}{Cx' + Dy'} + 2Bx' + B^2(Cx' + Dy')\right\}. \tag{16.11}$$

We start with the form (16.11) of the fundamental function F and firstly make use of the necessary condition (16.5). For the function (16.11) we have

$$F_1 = \frac{2}{(Cx' + Dy')^3}. \tag{16.12}$$

Consequently condition (16.5) runs as follows

$$\frac{\partial C}{\partial x}(x')^2 + \left(\frac{\partial C}{\partial y} + \frac{\partial D}{\partial x}\right)x'y' + \frac{\partial D}{\partial y}(y')^2 = 0. \tag{16.13}$$

Since C, D are functions of position, (16.13) drops into the system of differential equations

$$\frac{\partial C}{\partial x} = \frac{\partial C}{\partial y} + \frac{\partial D}{\partial x} = \frac{\partial D}{\partial y} = 0, \tag{16.14}$$

which gives, by integration,

$$C = -ay + c, \qquad D = ax + d \qquad (a,\, c,\, d \text{ constants}). \tag{16.15}$$

Now we distinguish two cases:
(a) $a = 0$.

$$C = c, \qquad D = d. \tag{16.15a}$$

(b) $a \neq 0$.

$$C = -a(y - y_0), \qquad D = a(x - x_0). \tag{16.15b}$$

In case (a) we make the transformation of coordinates

$$\bar{x} = x, \qquad \bar{y} = d^2(cx + dy) \tag{16.16a}$$

and put

$$\frac{1}{d^2} B(x, y) = \frac{1}{d^2} B\left(\bar{x}, \frac{\bar{y}}{d^3} - \frac{c}{d}\bar{x}\right) = z(\bar{x}, \bar{y}). \tag{16.17a}$$

By substituting in (16.11) and dropping the bars, we obtain

$$\boxed{F = \frac{(x')^2}{y'} + 2zx' + z^2 y' = \frac{(x' + zy')^2}{y'}}. \tag{16.18}$$

In case (b) the transformation of coordinates

$$\bar{x} = \frac{1}{x - x_0}, \qquad \bar{y} = a^3 \frac{y - y_0}{x - x_0} \tag{16.16b}$$

gives

$$C = -\frac{1}{a^2}\frac{\bar{y}}{\bar{x}}, \qquad D = a\frac{1}{\bar{x}}, \qquad x' = -\frac{\bar{x}'}{\bar{x}^2}, \qquad Cx' + Dy' = \frac{1}{a^2}\frac{\bar{y}'}{\bar{x}^2}.$$

Further we put

$$-\frac{1}{a^2} B(x, y) = -\frac{1}{a^2} B\left(\frac{1}{\bar{x}} + x_0, \frac{1}{a^3}\frac{\bar{y}}{\bar{x}} + y_0\right) = z(\bar{x}, \bar{y}). \tag{16.17b}$$

By substituting these values in (16.11) and dropping the bars, we recover (16.18).

The fundamental function (16.18) satisfies the necessary condition (16.5), but not yet the necessary and sufficient condition (16.4). This condition gives for z the differential equation

$$\boxed{z\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0.} \tag{16.19}$$

(16.19) has the obvious solution $z =$ const. which gives a Minkowski space. If z is not constant, integration of (16.19) leads to

$$\boxed{x + yz = \Psi(z).} \tag{16.20}$$

where Ψ is an arbitrary function of z (which we suppose to be analytic).

Hence we have:

The fundamental function of a two-dimensional Finsler space with rectilinear extremals and $I^2 = \frac{9}{2}$ *which is no Minkowski space, can be brought by a suitable transformation of coordinates, to the form* (16.18), *where* z *is a non constant solution of* (16.19). *These solutions are given by* (16.20), *where* $\Psi(z)$ *is an arbitrary function of* z.

In §17 we shall see that non constant solutions of (16.19) exist which nevertheless give a Minkowski space.

17. Space curvature in the exceptional case. We calculate now the curvature K of a Finsler space with rectilinear extremals and $I^2 = \frac{9}{2}$, using (11.1). The function

$$q = \frac{1}{F^2} l_i G^i \tag{17.1}$$

introduced by (9.2), is connected with F by

$$q = \frac{1}{2F^2} \frac{\partial F}{\partial x^i} x'^i \tag{17.2}$$

as is seen from

$$\frac{dF}{ds} = \frac{\partial F}{\partial x^i} \frac{x'^i}{F} - 2 \frac{1}{F} l_i G^i = 0. \tag{17.3}$$

Further we have from (9.2)

$$q_s = \frac{1}{F} \frac{\partial q}{\partial x^i} x'^i, \tag{17.4}$$

if we note the homogeneity of q in the x'.

Now let F be given by (16.18), (16.19). Then we substitute, with the aid of (16.19), the partial derivatives $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial^2 z}{\partial x^2}$ for the derivatives $\dfrac{\partial z}{\partial y}$ $\dfrac{\partial^2 z}{\partial y \partial x}$:

$$\frac{\partial z}{\partial y} = z \frac{\partial z}{\partial x}, \qquad \frac{\partial^2 z}{\partial y \partial x} = z \frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2. \tag{17.5}$$

Moreover it follows from (16.20), by partial differentiation with respect to x, that

$$\frac{\partial z}{\partial x} = \frac{1}{\Psi'(z) - y}, \qquad \frac{\partial^2 z}{\partial x^2} = -\frac{\Psi''(z)}{(\Psi'(z) - z)^3}, \tag{17.6}$$

where the primes indicate differentiation with respect to z. Thus we get

$$q = \left(\frac{y'}{x' + zy'}\right)^2 \frac{\partial z}{\partial x}, \qquad q_s = \left(\frac{y'}{x' + zy'}\right)^3 \frac{\partial^2 z}{\partial x^2} - \left(\frac{y'}{x' + zy'}\right)^4 \left(\frac{\partial z}{\partial x}\right)^2, \tag{17.7}$$

and by substituting these values in (11.1)

$$\boxed{K = \frac{\Psi''(z)}{(\Psi'(z) - y)^3} \left(\frac{y'}{x' + zy'}\right)^3}. \tag{17.8}$$

From (17.8) we see that K is then and only then a function of position only, when Ψ is linear in z. In this case we obtain from (16.20) $z = -\dfrac{x - x_0}{y - y_0}$ (x_0, y_0 constants). With regard to the theorem of §14 and to the remark on the case $z =$ const., made in §16, we have:

When, and only when, the function z which enters into the fundamental function (16.18), (16.19) *of a two-dimensional Finsler space with rectilinear extremals and* $I^2 = \frac{9}{2}$, *has one of the values*

$$z = z_0, \qquad z = -\frac{x - x_0}{y - y_0}, \qquad (x_0, y_0, z_0 \text{ constants})$$

the space is a Minkowski space. In every other case the curvature of the Finsler space depends on the line-element.

For $z = z_0$ the transformation of coordinates

$$\bar{x} = x + z_0 y, \qquad \bar{y} = y, \tag{17.9}$$

for $z = -\dfrac{x - x_0}{y - y_0}$ the transformation

$$\bar{x} = \frac{x - x_0}{y - y_0}, \qquad \bar{y} = -\frac{1}{y - y_0} \tag{17.10}$$

carries over the fundamental function in

$$F = \frac{(\bar{x}')^2}{\bar{y}'}. \tag{17.11}$$

18. Table of all two-dimensional Finsler spaces with rectilinear extremals the main scalar of which is a function of position. Now we are able to set up a table of all Finsler spaces with rectilinear extremals the main scalar I of which is a function of position only, and therefore (§15) constant. For $I^2 \neq 0, \frac{9}{2}$ these spaces are Minkowski spaces (§15). For $I = 0$, we have K constant (§15). Consequently the corresponding spaces are the Riemannian spaces of constant curvature. The case $I^2 = \frac{9}{2}$ was discussed in §§16, 17. With regard to (14.13), we obtain the following table:

(*I*.) $K =$ *const.*, $\neq 0$. $I = 0$. (*Non-euclidean spaces*).

(1) $K = \dfrac{1}{k^2} \quad : \quad F = k\,\dfrac{[(a^2 + y^2)(x')^2 - 2xyx'y' + (a^2 + x^2)(y')^2]^{\frac{1}{2}}}{a^2 + x^2 + y^2}$,

(2) $K = -\dfrac{1}{k^2} \quad : \quad F = k\,\dfrac{[(a^2 - y^2)(x')^2 + 2xyx'y' + (a^2 - x^2)(y')^2]^{\frac{1}{2}}}{a^2 - x^2 - y^2}$;

$(k > 0;\ a = \text{const.} > 0)$.

(*II*.) $K = 0$. $I =$ *const.* (*Euclidean space, respectively Minkowski spaces with* $I =$ *const.* $\neq 0$).

(3) $I^2 < 4$ $\quad : \quad F = [(x')^2 + (y')^2]^{\frac{1}{2}} e^{\frac{I}{\sqrt{(4-I)^2}} \operatorname{arctg} \frac{y'}{x'}}$,

(4) $I^2 = 4$ $\quad : \quad F = x' e^{\frac{y'}{x'}}$ [24]

(5) $I^2 > 4$ $\quad : \quad F = (x')^{\frac{1}{2}\left(1 - \frac{I}{\sqrt{(I^2-4)}}\right)} (y')^{\frac{1}{2}\left(1 + \frac{I}{\sqrt{(I^2-4)}}\right)}$.

(*III*). *K variable.* $I^2 = \frac{9}{2}$.

(6) $\quad F = \dfrac{(x' + zy')^2}{y'}, \quad x + yz = \Psi(z), \quad (\Psi$ arbitrary function, $\Psi''(z) \neq 0)$.

This table shows that the two-dimensional Finsler geometries with rectilinear extremals and constant curvature $K \neq 0$ which are not Riemannian (Hilbert's geometry and its generalization, Funk's geometry of the specific metric and so on),[25] have a main scalar which depends on the line-element $(x, y; x', y')$. The same is true for the two-dimensional Finsler geometries with rectilinear extremals and $K = 0$ which are not Minkowski geometries.[25]

Finally, let us remark that the spaces, enumerated in the above table under (*I.*) and (*II.*), admit a continuous three parameter group of transformations into themselves.[26]

19. Landsberg spaces with rectilinear extremals. A *Landsberg space* is a two-dimensional Finsler space with

$$I_{,s} = 0.^{27} \tag{19.1}$$

The Landsberg spaces with rectilinear extremals are characterized by (19.1), (11.4), (11.5). In consequence of (19.1), equation (11.4) becomes

$$I_{b\theta} = -2II_b\,. \tag{19.2}$$

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[24] First, we have, corresponding to $I = \pm 2$, the two fundamental functions $F = x'e^{\frac{y'}{x'}}$, $F^* = x'e^{-\frac{y'}{x'}}$. The transformation of coördinates $\bar{x} = x$, $\bar{y} = -y$ transforms F^* to F, if we drop the bars.

[25] P. Funk, [9], [10].

[26] For the spaces (II), (4), (5) cf. E. Nohel, [14], (especially groups 16, 20, 25), A. Maccone, [15], E. Cartan, [5], p. 135. In these papers, the space (II), (3) does not appear, because they take as starting point the complex domain, where the types (II), (3) and (II), (5) are not different. All spaces (II), (3)–(5) figure in S. Lie and F. Engel, [13], p. 435 ff.; but their ds is not there given.

[27] These spaces were first considered, from another point of view, by G. Landsberg, [12], p. 334 f. See also L. Berwald, [1], pp. 208 f., 211–213; E. Cartan, [5], p. 133 f.

From (19.1) and (19.2) we obtain, with the aid of the permutation formulas (7.6), the formulas

$$(19.3)\quad \begin{cases} I_{\vartheta b} = -II_b, & I_{\vartheta s} = -I_b, \\ I_{\vartheta\vartheta b} = (-I_\vartheta + I^2 - 1)I_b, & I_{\vartheta\vartheta s} = 3II_b, \\ I_{\vartheta\vartheta\vartheta b} = (-I_{\vartheta\vartheta} + 3II_\vartheta - I^3 + 4I)I_b, & I_{\vartheta\vartheta\vartheta s} = (4I_\vartheta - 7I^2 + 1)I_b, \\ I_{\vartheta\vartheta\vartheta\vartheta b} = (-I_{\vartheta\vartheta\vartheta} + 4II_{\vartheta\vartheta} + 3I_\vartheta^2 - 6I^2I_\vartheta + 8I_\vartheta + I^4 - 11I^2 + 1)I_b, & \end{cases}$$

which we shall use later on.

We desire to determine all Landsberg spaces with rectilinear extremals. For that purpose we state first the following theorem:

The Landsberg spaces, the extremals of which form a quasigeodesic system of curves (§12), *are identical with the two-dimensional affinely connected Finsler spaces.* (§8).

Proof: A Landsberg space with a quasigeodesic system of extremals is characterized by (19.1), (19.2). In consequence of (19.1) the Bianchi identity (7.8) reduces to

$$(19.4)\qquad K_\vartheta = -IK.$$

The third permutation formula (7.6) gives

$$(19.5)\qquad I_{bs} = KI_\vartheta .$$

Now apply the second formula (7.6) to $\Phi = I_b$. On account of (19.2), (19.5), (19.4), it follows that

$$(19.6)\qquad I_{bb} = I_{bs\vartheta} - I_{b\vartheta s} = K(I_{\vartheta\vartheta} + II_\vartheta).$$

Further we get from the first permutation formula (7.6)

$$I_{b\vartheta b} = I_{bb\vartheta} + I_{bs} + II_{bb} .$$

When we calculate both members by means of (19.2), (19.4), (19.6), we obtain

$$(19.7)\qquad 2I_b^2 + K(I_{\vartheta\vartheta\vartheta} + 3II_{\vartheta\vartheta} + I_\vartheta^2 + 2I^2I_\vartheta + I_\vartheta) = 0.$$

Differentiating (19.7) as to ϑ and using (19.2), (19.4), we find

$$(19.8)\qquad K[I_{\vartheta\vartheta\vartheta\vartheta} + 6II_{\vartheta\vartheta\vartheta} + (5I_\vartheta + 11I^2 + 1)I_{\vartheta\vartheta} + (7I_\vartheta + 6I^2 + 3)II_\vartheta] = 0.$$

If $K = 0$, it follows from (19.7) that $I_b = 0$. The space is a Minkowski space, because of $I_s = I_b = K = 0$ (§8).

Now let $K \neq 0$ and differentiate the second factor on the left of (19.8) with respect to b. With regard to (19.3), it follows that

$$(19.9)\qquad I_b(I_{\vartheta\vartheta\vartheta} + 3II_{\vartheta\vartheta} + I_\vartheta^2 + 2I^2I_\vartheta + I_\vartheta) = 0.$$

If $I_b = 0$, the space is an affinely connected space with $I =$ const., on account of (19.1) and $K \neq 0$ (§8). If

(19.10) $$I_{\vartheta\vartheta\vartheta} + 3II_{\vartheta\vartheta} + I_\vartheta^2 + 2I^2I_\vartheta + I_\vartheta = 0,$$

(19.7) gives again $I_b = 0$.

Conversely, it follows from (8.9) that for a two-dimensional affinely connected Finsler space (19.1), (19.2) are satisfied.

Now it is easy to find all Landsberg spaces with rectilinear extremals. We must only take, from the table of §18, the spaces with $K \neq 0$ and add all Minkowski spaces. Hence we have the theorem:

There exist but the following types of Landsberg spaces with rectilinear extremals:

(*I.*) $K = 0$: *Minkowski spaces.*

$$F = F(x', y').$$

(*II.*) $K =$ *const.*, $\neq 0$: *Non-euclidean spaces.*

$$(1) \quad K = \frac{1}{k^2} \quad : \quad F = k\,\frac{[(a^2 + y^2)(x')^2 - 2xyx'y' + (a^2 + x^2)(y')^2]^{\frac{1}{2}}}{a^2 + x^2 + y^2};$$

$$(2) \quad K = -\frac{1}{k^2} \quad : \quad F = k\,\frac{[(a^2 - y^2)(x')^2 + 2xyx'y' + (a^2 - x^2)(y')^2]^{\frac{1}{2}}}{a^2 - x^2 - y^2};$$

$$(k > 0,\ a = \text{const.},\ > 0).$$

(*III.*) *K variable.*

$$F = \frac{(x' + zy')^2}{y'}, \quad x + yz = \psi(z), \quad (\psi \text{ arbitrary function}, \psi''(z) \neq 0).$$

Prague, Bohemia.

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ON UNITARY REPRESENTATIONS OF THE GROUP OF DE SITTER SPACE

BY L. H. THOMAS

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The ten-parameter group of motions of De Sitter space is that of the real homogeneous linear transformations of w, x, y, z, and t, that leave $w^2 + x^2 + y^2 + z^2 - t^2$ invariant and that can be built up from infinitesimal transformations. It contains the sub-group of the real homogeneous linear transformations of w, x, y, and z, that leave $w^2 + x^2 + y^2 + z^2$ invariant and that can be built up from infinitesimal transformations, the six-parameter group of rotations of four-dimensional Euclidean space.

We seek differentiable representations by unitary matrices, in general infinite, of groups locally isomorphic with the ten-parameter group. Hermitian matrices are found for the infinitesimal transformations of representations which admit a transformation, perhaps to generalised matrices with a continuum of rows and columns, reducing the six-parameter sub-group to a product of its irreducible unitary representations.

1. INTRODUCTION

Wigner[1] has recently discussed the unitary representations of the inhomogeneous Lorentz group. If this group is regarded as the group of motions of a flat three plus one dimensional Riemannian space, it may be regarded as the limit for zero curvature of the group of motions of De Sitter space. Dirac[2] has considered the form that the electron wave equations take in De Sitter space. It is of some interest to discuss the unitary representations of the group of De Sitter space, which should be relevant to the quantum theory of the external properties of systems moving in such a space. These representations are investigated here by the standard methods of matrix-mechanics,[3] applied to the infinitesimal operators of the group, and it is believed that no differentiable representations admitting a transformation reducing the representation induced in the group of motions of a three-dimensional sub-space of constant positive curvature have been missed. The argument, however, lacks logical rigor in its application to the representations depending on continuous parameters.

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[1] *On Unitary Representations of the Inhomogeneous Lorentz Group.* Annals of Mathematics 40 (1939) p. 145.

[2] *The Electron Wave Equation in De Sitter Space.* Annals of Mathematics 36 (1935) p. 657.

[3] See e.g. Born and Jordan. *Elementare Quantenmechanik.* Ch. III. Springer (1930).

2. The Infinitesimal Operators of the Group and their Commutation Relations

$$L = -\iota\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \quad M = -\iota\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right), \quad N = -\iota\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right);$$

$$X = -\iota\left(w\frac{\partial}{\partial x} - x\frac{\partial}{\partial w}\right), \quad Y = -\iota\left(w\frac{\partial}{\partial y} - y\frac{\partial}{\partial w}\right), \quad Z = -\iota\left(w\frac{\partial}{\partial z} - z\frac{\partial}{\partial w}\right); \qquad 2.11$$

$$U = -\iota\left(t\frac{\partial}{\partial x} - x\frac{\partial}{\partial t}\right), \quad V = -\iota\left(t\frac{\partial}{\partial y} - y\frac{\partial}{\partial t}\right), \quad W = -\iota\left(t\frac{\partial}{\partial z} - z\frac{\partial}{\partial t}\right),$$

and

$$T = -\iota\left(t\frac{\partial}{\partial w} - w\frac{\partial}{\partial t}\right); \qquad 2.12$$

are a complete set of ten self-adjoint linear differential operators of the group, satisfying the 45 commutation relations:—

$$\begin{aligned}
&MN - NM = \iota L, \quad MZ - ZM = \iota X, \quad YN - NY = \iota X, \\
&\qquad LX - XL = 0, \quad YZ - ZY = \iota L, \\
&NL - LN = \iota M, \quad NX - XN = \iota Y, \quad ZL - LZ = \iota Y, \\
&\qquad MY - YM = 0, \quad ZX - XZ = \iota M, \\
&LM - ML = \iota N, \quad LY - YL = \iota Z, \quad XM - MX = \iota Z, \\
&\qquad NZ - ZN = 0, \quad XY - YX = \iota N; \\
&MW - WM = \iota U \quad VN - NV = \iota U, \quad LU - UL = 0, \\
&\qquad YW - WY = 0, \quad VZ - ZV = 0, \\
&NU - UN = \iota V, \quad WL - LW = \iota V, \quad MV - VM = 0, \\
&\qquad ZU - UZ = 0, \quad WX - XW = 0, \\
&LV - VL = \iota W, \quad UM - MU = \iota W, \quad NW - WN = 0, \\
&\qquad XV - VX = 0, \quad UY - YU = 0,
\end{aligned} \qquad 2.21$$

$$\begin{aligned}
UX - XU &= \iota T, \quad LT - TL = 0, \quad XT - TX = \iota U, \\
VY - YV &= \iota T, \quad MT - TM = 0, \quad YT - TY = \iota V, \\
WZ - ZW &= \iota T, \quad NT - TN = 0, \quad ZT - TZ = \iota W;
\end{aligned} \qquad 2.22$$

$$\begin{aligned} UT - TU &= \iota X, & VW - WV &= -\iota L, \\ VT - TV &= \iota Y, & WU - UW &= -\iota M, \\ WT - TW &= \iota Z, & UV - VU &= -\iota N. \end{aligned} \qquad 2.23$$

The first six of these operators (2.11) satisfying the first fifteen commutation relations (2.21) are a complete set of operators of the six-parameter sub-group.

To any differentiable representation of the ten-parameter group there must correspond a representation of these operators. If the representation of the group is unitary, that of these operators is Hermitian. A transformation reducing a representation of the sub-group to a sum of its irreducible unitary representations must reduce the representation of its operators to a sum of irreducible Hermitian representations.

The commutation relations of the displacement operators for free motion in quantum theory in De Sitter space can be reduced to the above form by change of phase.

3. Outline of the Method Employed

Let the (known) representations of the operators, $A = L, M, N, X, Y$, and Z, of the sub-group be expressed in the form (in Dirac's notation)

$$(\alpha \mid A(\beta) \mid \alpha') \qquad 3.1$$

where β is a set of parameters taking a denumerable infinity of discrete values, one for each irreducible representation, and α, α', are sets of parameters numbering the columns and rows of the matrices, and taking for each β only a finite number of discrete values. Then a transformation reducing a representation of the sub-group to a sum of these representations reduces it to the form

$$\delta_{\gamma\gamma'}\delta_{\beta\beta'} \quad (\alpha \mid A(\beta) \mid \alpha') \qquad 3.2$$

where $\delta_{\beta\beta'}$ is the usual Kronecker δ; γ, γ', are sets of parameters numbering the columns and rows of the (generalised) matrices, taking for each β, perhaps a continuous, perhaps a discrete set of values, perhaps no values at all, and $\delta_{\gamma\gamma'}$ is to be interpreted accordingly.

If the remaining operators, $P = U, V, W$, and T, of the group admit the transformation, they take the form

$$(\gamma, \beta, \alpha \mid P \mid \gamma', \beta', \alpha'). \qquad 3.3$$

The 24 commutation relations of the P's with the A's (2.22) enable us to express the P's in the form

$$(\gamma \mid P(\beta, \beta') \mid \gamma')(\alpha \mid P(\beta, \beta') \mid \alpha') \qquad 3.4$$

where $(\alpha \mid P(\beta, \beta') \mid \alpha')$ are determined, and indeed for any β fail to vanish only for at most four values of β'.

In virtue of the fifteen commutation relations of the A's (2.21) and the 24 commutation relations of the P's with the A's (2.22), the remaining six commuta-

tion relations of the P's (2.23) follow from any one of them. This one then determines possible forms for $(\gamma \mid P(\beta, \beta') \mid \gamma')$.

4. The Representations of the Sub-group

The work is shortened if the sub-group is expressed as the direct product of two three-parameter groups.

Let

$$\begin{aligned} A_1 &= L + X, & A_2 &= L - X, \\ B_1 &= M + Y, & B_2 &= M - Y, \\ C_1 &= N + Z, & C_2 &= N - Z, \end{aligned} \tag{4.1}$$

so that any one of A_1, B_1, and C_1, commutes with any one of A_2, B_2, and C_2, and these are the operators of factor groups for either of which

$$\begin{aligned} BC - CB &= 2\iota A, \\ CA - AC &= 2\iota B, \\ AB - BA &= 2\iota C. \end{aligned} \tag{4.2}$$

If

$$\begin{aligned} A - \iota B &= C_-, \\ A + \iota B &= C_+, \end{aligned}$$

we have

$$\begin{aligned} CC_- - C_-C &= -2C_-, \\ CC_+ - C_+C &= 2C_+, \\ C_-C_+ - C_+C_- &= -4C, \end{aligned} \tag{4.3}$$

and the only non-zero components of an irreducible (Hermitian) representation may be brought to the form

$$\begin{aligned} (m \mid C_- \mid m + 1) &= 2\sqrt{\{(j - m)(j + m + 1)\}}, \\ & \qquad m = -j, -j + 1, \cdots, j - 1, j, \\ (m \mid C_+ \mid m - 1) &= 2\sqrt{\{(j + m)(j - m + 1)\}}, \\ (m \mid C \mid m) &= 2m, \end{aligned} \tag{4.4}$$

where for different representations

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots . \tag{4.5}$$

Thus we may take for β above j_1 and j_2 for these factor groups, taking any values from (4.5), and for α above m_1 and m_2 satisfying (4.4), and the matrices representing L, M, N, X, Y, and Z, may be written down immediately, in particular

$$(m_1, m_2 \mid Z(j_1, j_2) \mid m_1, m_2) = m_1 - m_2 \tag{4.6}$$

for $m_1 = -j_1, -j_1 + 1, \cdots, j_1$, $m_2 = -j_2, -j_2 + 1, \cdots, j_2$, the remaining components vanishing.

5. The Determination of $(\alpha \mid P(\beta, \beta') \mid \alpha')$

The 24 commutation relations (2.22) give for the operators of the first factor group

$$\begin{aligned}(T + \iota W)C - C(T + \iota W) &= -(T + \iota W),\\ (U - \iota V)C - C(U - \iota V) &= (U - \iota V),\end{aligned} \tag{5.11}$$

$$\begin{aligned}(T + \iota W)C_+ - C_+(T + \iota W) &= 0,\\ (U - \iota V)C_- - C_-(U - \iota V) &= 0,\end{aligned} \tag{5.12}$$

$$\begin{aligned}(T + \iota W)C_- - C_-(T + \iota W) &= -2\iota(U - \iota V),\\ (U - \iota V)C_+ - C_+(U - \iota V) &= 2\iota(U - \iota V),\end{aligned} \tag{5.13}$$

and similar equations for $T - \iota W$ and $U - \iota V$ with C_+ and C_- interchanged and the sign of C reversed; and for the second factor group similar equations with the sign of T reversed.

For (5.11) to hold it is necessary and sufficient that a component of the transformed representation of $T + \iota W$ vanish unless $m' = m - \frac{1}{2}$, one of $U - \iota V$ unless $m' = m + \frac{1}{2}$, and then for (5.12) to hold that in these cases they involve m only in factors $\sqrt{\{(j + m)!(j' - m + \frac{1}{2})!/(j - m)!/(j' + m - \frac{1}{2})!\}}$ and $\sqrt{\{(j - m)!(j' + m + \frac{1}{2})!/(j + m)!(j' - m - \frac{1}{2})!\}}$ respectively. (5.13) gives then

$$\{(j - j')^2 - \tfrac{1}{4}\}\{(j + j' + 1)^2 - \tfrac{1}{4}\} = 0,$$

so that the components vanish unless $j' = j - \frac{1}{2}$ or $j' = j + \frac{1}{2}$, and we must have for the only non-zero components,

$$\begin{aligned}(\lambda, j_1, m_1 \mid T + \iota W \mid \lambda', j_1 - \tfrac{1}{2}, m_1 - \tfrac{1}{2}) &= (\lambda \mid u(j_1) \mid \lambda')\sqrt{(j_1 + m_1)},\\ (\lambda, j_1, m_1 \mid T + \iota W \mid \lambda', j_1 + \tfrac{1}{2}, m_1 - \tfrac{1}{2}) &= (\lambda \mid v(j_1) \mid \lambda')\sqrt{(j_1 - m_1 + 1)},\\ (\lambda, j_1, m_1 \mid U - \iota V \mid \lambda', j_1 - \tfrac{1}{2}, m_1 + \tfrac{1}{2}) &= -\iota(\lambda \mid u(j_1) \mid \lambda')\sqrt{(j_1 - m_1)},\\ (\lambda, j_1, m_1 \mid U - \iota V \mid \lambda', j_1 + \tfrac{1}{2}, m_1 + \tfrac{1}{2}) &= \iota(\lambda \mid v(j_1) \mid \lambda')\sqrt{(j_1 + m_1 + 1)},\end{aligned}$$

where λ includes γ, j_2, and m_2; necessary and sufficient conditions that (5.11), (5.12), and (5.13), be satisfied, $u(j_1)$ and $v(j_1)$ being arbitrary matrix functions of j_1 for $j_1 = \frac{1}{2}, 1, \frac{3}{2}, \cdots$, and for $j_1 = 0, \frac{1}{2}, 1, \cdots$, respectively.

Similarly, for their only non-zero components,

$$\begin{aligned}(\lambda, j_1, m_1 \mid T - \iota W \mid \lambda', j_1 + \tfrac{1}{2}, m_1 + \tfrac{1}{2}) &= (\lambda \mid u'(j_1) \mid \lambda')\sqrt{(j_1 + m_1 + 1)},\\ (\lambda, j_1, m_1 \mid T - \iota W \mid \lambda', j_1 - \tfrac{1}{2}, m_1 + \tfrac{1}{2}) &= (\lambda \mid v'(j_1) \mid \lambda')\sqrt{(j_1 - m_1)},\\ (\lambda, j_1, m_1 \mid U + \iota V \mid \lambda', j_1 + \tfrac{1}{2}, m_1 - \tfrac{1}{2}) &= \iota(\lambda \mid u'(j_1) \mid \lambda')\sqrt{(j_1 - m_1 + 1)},\\ (\lambda, j_1, m_1 \mid U + \iota V \mid \lambda', j_1 - \tfrac{1}{2}, m_1 - \tfrac{1}{2}) &= -\iota(\lambda \mid v'(j_1) \mid \lambda')\sqrt{(j_1 + m_1)},\end{aligned}$$

where $u'(j_1)$ and $v'(j_1)$ are further arbitrary matrix functions of j_1 for $j_1 = 0, \frac{1}{2}, 1, \cdots$, and for $j_1 = \frac{1}{2}, 1, \frac{3}{2}, \cdots$, respectively.

Combining these results with the corresponding results for the second factor group, for which the components of $T + \iota W$ are connected with those of $U + \iota V$, and those of $T - \iota W$ with those of $U - \iota V$, we obtain the necessary and sufficient conditions that the 24 commutation relations (2.22) be satisfied with L, M, N, X, Y, and Z, given as in (4.). The only non-zero components of $T + \iota W$, $U - \iota V$, $U + \iota V$, and $T - \iota W$, have the form

$$
\begin{aligned}
&(\gamma, j_1, m_1, j_2, m_2 \mid T + \iota W \mid \gamma', j_1 - \tfrac{1}{2}, m_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}, m_2 + \tfrac{1}{2}) \\
&\qquad = (\gamma \mid f(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 + m_1)(j_2 - m_2)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid T + \iota W \mid \gamma', j_1 + \tfrac{1}{2}, m_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}, m_2 + \tfrac{1}{2}) \\
&\qquad = (\gamma \mid g(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 - m_1 + 1)(j_2 - m_2)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid T + \iota W \mid \gamma', j_1 - \tfrac{1}{2}, m_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}, m_2 + \tfrac{1}{2}) \\
&\qquad = (\gamma \mid h(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 + m_1)(j_2 + m_2 + 1)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid T + \iota W \mid \gamma', j_1 + \tfrac{1}{2}, m_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}, m_2 + \tfrac{1}{2}) \\
&\qquad = (\gamma \mid k(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 - m_1 + 1)(j_2 + m_2 + 1)\}}.
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
&(\gamma, j_1, m_1, j_2, m_2 \mid U - \iota V \mid \gamma', j_1 - \tfrac{1}{2}, m_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}, m_2 + \tfrac{1}{2}) \\
&\qquad = -\iota(\gamma \mid f(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 - m_1)(j_2 - m_2)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid U - \iota V \mid \gamma', j_1 + \tfrac{1}{2}, m_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}, m_2 + \tfrac{1}{2}) \\
&\qquad = \iota(\gamma \mid g(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 + m_1 + 1)(j_2 - m_2)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid U - \iota V \mid \gamma', j_1 - \tfrac{1}{2}, m_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}, m_2 + \tfrac{1}{2}) \\
&\qquad = -\iota(\gamma \mid h(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 - m_1)(j_2 + m_2 + 1)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid U - \iota V \mid \gamma', j_1 + \tfrac{1}{2}, m_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}, m_2 + \tfrac{1}{2}) \\
&\qquad = \iota(\gamma \mid k(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 + m_1 + 1)(j_2 + m_2 + 1)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid U + \iota V \mid \gamma', j_1 - \tfrac{1}{2}, m_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}, m_2 - \tfrac{1}{2}) \\
&\qquad = \iota(\gamma \mid f(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 + m_1)(j_2 + m_2)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid U + \iota V \mid \gamma', j_1 + \tfrac{1}{2}, m_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}, m_2 - \tfrac{1}{2}) \\
&\qquad = \iota(\gamma \mid g(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 - m_1 + 1)(j_2 + m_2)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid U + \iota V \mid \gamma', j_1 - \tfrac{1}{2}, m_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}, m_2 - \tfrac{1}{2}) \\
&\qquad = -\iota(\gamma \mid h(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 + m_1)(j_2 - m_2 + 1)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid U + \iota V \mid \gamma', j_1 + \tfrac{1}{2}, m_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}, m_2 - \tfrac{1}{2}) \\
&\qquad = -\iota(\gamma \mid k(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 - m_1 + 1)(j_2 - m_2 + 1)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid T - \iota W \mid \gamma', j_1 - \tfrac{1}{2}, m_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}, m_2 - \tfrac{1}{2}) \\
&\qquad = -(\gamma \mid f(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 - m_1)(j_2 + m_2)\}}, \\
&(\gamma, j_1, m_1, j_2, m_2 \mid T - \iota W \mid \gamma', j_1 + \tfrac{1}{2}, m_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}, m_2 - \tfrac{1}{2}) \\
&\qquad = (\gamma \mid g(j_1, j_2) \mid \gamma')\sqrt{\{(j_1 + m_1 + 1)(j_2 + m_2)\}},
\end{aligned}
$$

$$(\gamma, j_1, m_1, j_2, m_2 \mid T - \iota W \mid \gamma', j_1 - \tfrac{1}{2}, m_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}, m_2 - \tfrac{1}{2})$$
$$= (\gamma \mid h(j_1, j_2) \mid \gamma' \sqrt{\{(j_1 - m_1)(j_2 - m_2 + 1)\}},$$
$$(\gamma, j_1, m_1, j_2, m_2 \mid T - \iota W \mid \gamma', j_1 + \tfrac{1}{2}, m_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}, m_2 - \tfrac{1}{2})$$
$$= -(\gamma \mid k(j_1, j_2) \mid \gamma') \sqrt{\{(j_1 + m_1 + 1)(j_2 - m_2 + 1)\}},$$

where $f(j_1, j_2)$, $g(j_1, j_2)$, $h(j_1, j_2)$, and $k(j_1, j_2)$, are in general arbitrary matrix functions of j_1 and j_2, j_1 taking values from $\frac{1}{2}$, 1, $\frac{3}{2}$, $\cdots$, in f and h, from 0, $\frac{1}{2}$, 1, $\cdots$, in g and k, j_2 taking values from $\frac{1}{2}$, 1, $\frac{3}{2}$, $\cdots$, in f and g, from 0, $\frac{1}{2}$, 1, $\cdots$, in h and k, but not necessarily all such pairs of values.

If the representation is Hermitian, $f(j_1, j_2)$ and $-k(j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$ must be Hermitian conjugates, and $g(j_1, j_2)$ and $h(j_1 + \frac{1}{2}, j_2 - \frac{1}{2})$ must be Hermitian conjugates. 5.3

It should be noted that the factors involving m_1 and m_2 make components connecting allowed values of m_1 and m_2 to those that are not allowed vanish, so that we need not usually take explicit consideration of the ranges of m_1 and m_2.

6. The General Form of f, g, h, and k

The remaining six commutation relations now reduce to any one of them; in particular $WT - TW = \iota Z$ in the form

$$(T + \iota W)(T - \iota W) - (T - \iota W)(T + \iota W) = -2Z$$

gives

$$\begin{aligned} j_1 f(j_1, j_2) g(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}) &= (j_1 + 1) g(j_1, j_2) f(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}), \\ j_2 f(j_1, j_2) h(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}) &= (j_2 + 1) h(j_1, j_2) f(j_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}), \\ j_2 g(j_1, j_2) k(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}) &= (j_2 + 1) k(j_1, j_2) g(j_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}), \\ j_1 h(j_1, j_2) k(j_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}) &= (j_1 + 1) k(j_1, j_2) h(j_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}), \end{aligned} \qquad 6.11$$

for $j_1' = j_1$, $j_2' = j_2 - 1$; $j_1' = j_1 - 1$, $j_2' = j_2$; $j_1' = j_1 + 1$, $j_2' = j_2$; and $j_1' = j_1$, $j_2' = j_2 + 1$; respectively in β', β: and

$$\begin{aligned} &(m_1 j_2 - m_2 j_1) f(j_1, j_2) k(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}) \\ &+ (m_1 j_2 + m_2(j_1 + 1)) g(j_1, j_2) h(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}) \\ &- (m_1(j_2 + 1) + m_2 j_1) h(j_1, j_2) g(j_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}) \\ &- (m_1(j_2 + 1) - m_2(j_1 + 1)) k(j_1, j_2) f(j_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}) = m_1 - m_2, \end{aligned} \qquad 6.12$$

for $j_1' = j_1$, $j_2' = j_2$: all the other components vanishing identically.

These relations hold whenever both j_1, j_2, and j_1', j_2', are possible values of β, non-existent terms being interpreted as vanishing.

Irreducible solutions of these equations are of two types; those for which the least value of either j_1 or j_2 in β in any non-vanishing component of (3.3) differs from zero; and those in which both these values are zero.

For the first type suppose, for example, $j_1 \nless p > 0$, and $j_1 = p$, for $j_2 = q$. Then (6.12) gives two independent relations between non-vanishing components of $g(p, q)h(p + \frac{1}{2}, q - \frac{1}{2})$ and $k(p, q)f(p + \frac{1}{2}, q + \frac{1}{2})$, which are

therefore constant multiples of an idempotent matrix, and $g(p, q)$ commutes with $h(p + \frac{1}{2}, q - \frac{1}{2})$, and $k(p, q)$ with $f(p + \frac{1}{2}, q + \frac{1}{2})$. Equations (6.11) then show that $g(p, q + 1)h(p + \frac{1}{2}, q + \frac{1}{2})$, $k(p, q - 1)f(p, q - \frac{1}{2})$, $g(p + \frac{1}{2}, q + \frac{1}{2})h(p + 1, q)$, and $k(p + \frac{1}{2}, q - \frac{1}{2})f(p + 1, q)$, are sums of multiples, zero for the first two, of the same idempotent matrix and of matrices whose product with it vanishes. Equations (6.12) then give similar information about $g(p + \frac{1}{2}, q - \frac{1}{2})h(p + 1, q - 1)$, $g(p + 1, q)h(p + \frac{3}{2}, q - \frac{1}{2})$, $k(p + 1, q)f(p + \frac{3}{2}, q + \frac{1}{2})$, and $k(p + \frac{1}{2}, q + \frac{1}{2})f(p + 1, q + 1)$; and so on. There are more than sufficient equations. If inconsistent they show that no such representation exists; if consistent that it decomposes into a term for which, for non-vanishing components, $j_1 > p$ for $j_2 = q$ which can be treated in the same way, and a term for which each $g(j_1, j_2)h(j_1 + \frac{1}{2}, j_2 - \frac{1}{2})$ and $k(j_1, j_2)f(j_1 + \frac{1}{2}, j_2 + \frac{1}{2})$ is a determined multiple of the same idempotent matrix.

This result and equations (6.11) now show that different transformations among γ, γ', for various j_1, j_2, can be taken so as to make all of f, g, h, and k, simultaneously diagonal, and the representation is reduced to a sum of representations in which f, g, h, and k, have but one row and column; if the original representation was Hermitian, the transformations required form a unitary transformation.

For the second type suppose $j_1 = 0$ for $j_2 = q$.

(6.12) now gives only one relation between $g(0, q)h(\frac{1}{2}, q - \frac{1}{2})$ and $k(0, q)f(\frac{1}{2}, q + \frac{1}{2})$; if we make one of these diagonal, so must be the other, and the argument for the previous case, introducing at each stage different transformations among γ, γ', for various j_1, j_2, shows that either no representation exists, or that it decomposes into a term for which always $j_1 > 0$, for $j_2 = q$, and a sum of terms for each of which each $g(j_1, j_2)h(j_1 + \frac{1}{2}, j_2 - \frac{1}{2})$ and $k(j_1, j_2)f(j_1 + \frac{1}{2}, j_2 + \frac{1}{2})$ is a determined multiple of an idempotent matrix, the products of the several idempotent matrices vanishing; perhaps continuously many such terms corresponding to possibly continuously many characteristic values of $g(0, q)h(\frac{1}{2}, q - \frac{1}{2})$. Finally as in the previous case, the representation reduces to a sum of representations in which f, g, h, and k, have but one row and column. It is this argument, involving matrices with continuously many rows and columns, which lacks logical rigor.

Thus for irreducible representations, $f(j_1, j_2)$, $g(j_1, j_2)$, $h(j_1, j_2)$, and $k(j_1, j_2)$, in (5.1) can be treated as numbers, $f(j_1, j_2)$, $g(j_1, j_2)$, $h(j_1, j_2)$, and $k(j_1, j_2)$. 6.2

Moreover the relations (6.11) and (6.12) involve either only values of β for which $j_1 + j_2$ is an integer or only values for which it is half an odd integer, so that any irreducible representation involves either only integral values of $j_1 + j_2$ or only half odd integral values. 6.3

7. The Solution of the Recurrence Relations

Equations (6.11), if $g(j_1, j_2)g(j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$ and $h(j_1, j_2 - 1)h(j_1 + \frac{1}{2}, j_2 - \frac{1}{2})$ do not both vanish, give

$$j_1(j_2 + \tfrac{1}{2})f(j_1, j_2)k(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2})$$
$$= (j_1 + 1)(j_2 - \tfrac{1}{2})f(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2})k(j_1, j_2 - 1),$$

so that in general

$$f(j_1, j_2)k(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}) = -\frac{F(j_1 + j_2)}{j_1(j_1 + \frac{1}{2})j_2(j_2 + \frac{1}{2})}. \qquad 7.11$$

Likewise, unless $f(j_1, j_2)f(j_1 + \frac{1}{2}, j_2 - \frac{1}{2})$ and $k(j_1 - \frac{1}{2}, j_2 - \frac{1}{2})k(j_1, j_2 - 1)$ both vanish for values of j_1 and j_2 involved,

$$g(j_1, j_2)h(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}) = \frac{G(j_1 - j_2)}{(j_1 + \frac{1}{2})(j_1 + 1)j_2(j_2 + \frac{1}{2})} \qquad 7.12$$

Then (6.12) gives

$$\frac{(m_1 j_2 - m_2 j_1)F(j_1 + j_2)}{j_1(j_1 + \frac{1}{2})j_2(j_2 + \frac{1}{2})} - \frac{(m_1(j_2 + 1) - m_2(j_1 + 1))F(j_1 + j_2 + 1)}{(j_1 + \frac{1}{2})(j_1 + 1)(j_2 + \frac{1}{2})(j_2 + 1)}$$

$$- \frac{(m_1 j_2 + m_2(j_1 + 1))G(j_1 - j_2)}{(j_1 + \frac{1}{2})(j_1 + 1)j_2(j_2 + \frac{1}{2})} + \frac{(m_1(j_2 + 1) - m_2(j_1 + 1))G(j_1 + j_2 + 1)}{j_1(j_1 + \frac{1}{2})(j_2 + \frac{1}{2})(j_2 + 1)} \qquad 7.2$$

$$= m_2 - m_1$$

Putting $m_2 = m_1$ in (7.2) we obtain

$$\frac{F(j_1 + j_2) + G(j_1 + j_2)}{j_2(j_1 + \frac{1}{2})} + \frac{F(j_1 + j_2 + 1) + G(j_1 - j_2 - 1)}{(j_2 + 1)(j_1 + \frac{1}{2})}$$

$$= \frac{F(j_1 + j_2) + G(j_1 - j_2 - 1)}{j_1(j_2 + \frac{1}{2})} + \frac{F(j_1 + j_2 + 1) + G(j_1 - j_2)}{(j_1 + 1)(j_2 + \frac{1}{2})},$$

from which follows

$$F(j_1 + j_2) + G(j_1 - j_2) = j_2(j_1 + \tfrac{1}{2})[\varphi(j_1 + j_2) + \psi(j_1 - j_2)]. \qquad 7.3$$

Subtracting this equation from the corresponding equation with j_1 and j_2 replaced by $j_1 + \frac{1}{2}$ and $j_2 + \frac{1}{2}$ we see that ψ is a quadratic function. Likewise φ is a quadratic function, so that F and G are of the fourth degree, and for (7.3) to hold, of the forms

$$F(j_1 + j_2) = \quad a[(j_1 + j_2)^2 + (j_1 + j_2)]^2 + b[(j_1 + j_2)^2 + (j_1 + j_2)] + c,$$

$$G(j_1 - j_2) = -a[(j_1 - j_2)^2 + (j_1 - j_2)]^2 - b[(j_1 - j_2)^2 + (j_1 - j_2)] - c.$$

Substituting these values, (7.2) is satisfied identically if $a = 1/16$, and we have

$$-f(j_1, j_2)k(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2})$$

$$= \frac{(j_1 + j_2 - p)(j_1 + j_2 + p + 1)(j_1 + j_2 - q)(j_1 + j_2 + q + 1)}{2j_1(2j_1 + 1)2j_2(2j_2 + 1)}, \qquad 7.41$$

$$g(j_1, j_2)k(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2})$$

$$= -\frac{(j_1 - j_2 - p)(j_1 - j_2 + p + 1)(j_1 - j_2 - q)(j_1 - j_2 + q + 1)}{(2j_1 + 1)(2j_1 + 2)2j_2(2j_2 + 1)}, \qquad 7.42$$

where p and q are arbitrary.

Starting with any given pair of values of j_1 and j_2 from $0, \frac{1}{2}, \frac{3}{2}, \cdots$, for which these products do not vanish, we may use the recurrence relations to find the values of these products for adjoining pairs of such values for which the sum of j_1 and j_2 has the initial form (either an integer or half an odd integer) unless stopped by zero factors in the numerators, which must occur before reaching pairs of values outside the above range. In this case the separate factors of these products can always be chosen so that all the equations (6.11) are satisfied, and (5.1) yield matrices satisfying all the commutation relations. 7.5

In the special cases in which either all the products $f(j_1, j_2)k(j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$ or all the products $g(j_1, j_2)h(j_1 + \frac{1}{2}, j_2 - \frac{1}{2})$ vanish, it can be verified that the surviving recurrence relations suffice to require the same form (7.41) or (7.42) for the surviving products.

8. Invariants

For matrices of the above form

$$T^2 + U^2 + V^2 + W^2 - (L^2 + M^2 + N^2 + X^2 + Y^2 + Z^2) \qquad 8.1$$

must exist, and, commuting with each matrix, be an invariant.

Writing it in the form

$$\begin{aligned}\tfrac{1}{2}[(T + \iota W)(T - \iota W) + (T - \iota W)(T + \iota W)]& \\ + \tfrac{1}{2}[(U + \iota V)(U - \iota V) + (U - \iota V)(U + \iota V)]& \\ - \tfrac{1}{2}[(L + X)^2 + (M + Y)^2 + (N + Z)^2]& \\ - \tfrac{1}{2}[(L - X)^2 + (M - Y)^2 + (N - Z)^2]&,\end{aligned}$$

we see that unless $\alpha' = \alpha$ and $\beta' = \beta$, its components vanish, and that for β and β' given by j_1, j_2, the (diagonal) component has the form

$$\begin{aligned}-2j_1j_2f(j_1, j_2)k(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}) + 2(j_1 + 1)j_2g(j_1, j_2)h(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2})& \\ + 2j_1(j_2 + 1)h(j_1, j_2)g(j_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2})& \\ - 2(j_1 + 1)(j_2 + 1)k(j_1, j_2)f(j_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2})& \qquad 8.11 \\ - 2j_1(j_1 + 1) - 2j_2(j_2 + 1)& \\ = \tfrac{5}{2} - (q + \tfrac{1}{2})^2 - (p + \tfrac{1}{2})^2&,\end{aligned}$$

which must be its value for a representation determined by q and p.

Again

$$\begin{aligned}(LX + MY + NZ)^2 - (LU + MV + NW)^2 - (LT + WY - UZ)^2& \\ - (MT + UZ - WX)^2 - (NT + VX - UY)^2& \qquad 8.2\end{aligned}$$

must exist, and, commuting with each matrix, be an invariant.

Writing this in the form

$$K^2 + (KT - TK)^2 + (KU - UK)^2 + (KV - VK)^2 + (KW - WK)^2,$$

where $K = LX + MY + NZ$, has only diagonal components $j_1(j_1 + 1) - j_2(j_2 + 1)$, so that we can at once write down the components of $K(T + \iota W) - (T + \iota W)K$, etc., we see that unless $\alpha = \alpha'$ and $\beta = \beta'$, the components of (8.2) vanish, and that for β and β' given by j_1, j_2, the diagonal component has the form

$$\begin{aligned}(j_1(j_1 + 1) - j_2(j_2 + 1))^2 &+ 2j_1j_2(j_1 - j_2)^2 f(j_1, j_2)k(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}) \\ &- 2(j_1 + 1)j_2(j_1 + j_2 + 1)^2 g(j_1, j_2)h(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}) \\ &- 2j_1(j_2 + 1)(j_1 + j_2 + 1)^2 h(j_1, j_2)g(j_1 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}) \\ &+ 2(j_1 + 1)(j_2 + 1)(j_1 - j_2)^2 k(j_1, j_2)f(j_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}) \\ &= p(p + 1)q(q + 1),\end{aligned} \tag{8.21}$$

which must be its value for a representation determined by q and p.

These results enable us to show that representations for distinct pairs of values of q and p are distinct. (In the quantum theory of the free motion of a system in De Sitter space, (8.1) would be proportional to the square of the rest mass, (8.2) for given mass, to the negative square of the spin.)

9. The Allowed Values of p and q

In order that the matrices should be Hermitian, $f(j_1, j_2)$ and $-k(j_1 - \frac{1}{2}, j_2 - \frac{1}{2})$ must be complex conjugates and $g(j_1, j_2)$ and $h(j_1 + \frac{1}{2}, j_2 + \frac{1}{2})$ must be complex conjugates, (5.3), (6.2). Thus for all pairs of values taken by j_1, j_2, we must have

$$\begin{aligned}-f(j_1, j_2)k(j_1 - \tfrac{1}{2}, j_2 - \tfrac{1}{2}) &\nless 0, \\ g(j_1, j_2)h(j_1 + \tfrac{1}{2}, j_2 - \tfrac{1}{2}) &\nless 0.\end{aligned} \tag{9.1}$$

Then by altering the phases, we can make

$$\begin{aligned}f(j_1, j_2) &= \iota\sqrt{\left\{\frac{(j_1 + j_2 - p)(j_1 + j_2 + p + 1)(j_1 + j_2 - q)(j_1 + j_2 + q + 1)}{2j_1(2j_1 + 1)2j_2(2j_2 + 1)}\right\}}, \\ g(j_1, j_2) &= \sqrt{\left\{-\frac{(j_1 - j_2 - p)(j_1 - j_2 + p + 1)(j_1 - j_2 - q)(j_1 - j_2 + q + 1)}{(2j_1 + 1)(2j_2 + 2)2j_2(2j_2 + 1)}\right\}}, \\ h(j_1, j_2) &= \sqrt{\left\{-\frac{(j_1 - j_2 - p - 1)(j_1 - j_2 + p)(j_1 - j_2 - q - 1)(j_1 - j_2 + q)}{2j_1(2j_1 + 1)(2j_2 + 1)(2j_2 + 2)}\right\}}, \\ k(j_1, j_2) &= \iota\sqrt{\left\{\frac{(j_1 + j_2 - p + 1)(j_1 + j_2 + p + 2)(j_1 + j_2 - q + 1)(j_1 + j_2 + q + 2)}{(2j_1 + 1)(2j_1 + 2)(2j_2 + 1)(2j_2 + 2)}\right\}},\end{aligned} \tag{9.2}$$

where the radicals are taken real and positive or zero.

Either I only integral values of $j_1 + j_2$, and therefore also of $j_1 - j_2$, are involved, or II only half odd integral values, (6.3).

I. For factors in the numerators to vanish (7.5), at least one of p, q, $-p - 1$, or $-q - 1$, must be a positive integer or zero; since interchange of p and q,

or of p and $-p-1$ does not affect (9.2), we may suppose $p \nless 0$ is an integer. Then $(q+\frac{1}{2})^2$ must be real for the expressions under the radical signs to be real.

IA. Suppose q is not an integer. Since a zero factor in the numerator of $f(j_1, j_2)$ must occur at the lower limit of $j_1 + j_2$ which must be not less than zero, $j_1 + j_2$ must take in $f(j_1, j_2)$ the values $p+1, p+2, \cdots$, and likewise in $k(j_1, j_2)$ the values $p, p+1, \cdots$, which must make the radicals real and positive.

Thus

$$(p+\tfrac{3}{2})^2 > (q+\tfrac{1}{2})^2. \qquad 9.3$$

$j_1 + j_2$ then takes in $g(j_1, j_2)$ and $h(j_1, j_2)$ the values $p, p+1, \cdots$.

Since a zero factor in the numerator of $g(j_1, j_2)$ must occur, for any allowed value of $j_1 + j_2$, in particular $j_1 + j_2 = p$, at lower and upper limits of $j_1 - j_2$ which must make both j_1 and j_2 positive or zero, $j_1 - j_2$ must take in $g(j_1, j_2)$ the values $-p, -p+1, \cdots, p-1$, and likewise in $h(j_1, j_2)$ the values $-p+1, -p+2, \cdots, p$, and in $f(j_1, j_2)$ and $k(j_1, j_2)$ the values $-p, -p+1, \cdots, p$.

For $p > 0$, therefore, $j_1 - j_2 = 0$ being a possible value in $g(j_1, j_2)$ or $h(j_1, j_2)$,

$$(\tfrac{1}{2})^2 > (q+\tfrac{1}{2})^2, \qquad 9.41$$

which in fact implies (9.3).

For the special case $p = 0$, this last condition is not necessary, since $g(j_1, j_2)$ and $h(j_1, j_2)$ do not take any values, and we have from (9.3) merely

$$(\tfrac{3}{2})^2 > (q+\tfrac{1}{2})^2. \qquad 9.42$$

The only integral value of q within these limits is $q = 0$ for $p = 0$.

IB. Suppose q is an integer, which, on account of the interchangeability of q and $-q-1$ and of p and q may be taken to be such that $q \nless p \nless 0$.

For $q = p$, the argument of (IA) applies, and the only possible values are $p = 0$, $q = 0$, which can be regarded as a special case of (IA).

For $q > p > 0$, since $j_1 + j_2$ can take only values which make the radicals in $f(j_1, j_2)$ and $k(j_1, j_2)$ real, $j_1 + j_2$ must take the values $q+1, q+2, \cdots$, in $f(j_1, j_2)$ and $q, q+1, \cdots$, in $k(j_1, j_2)$, and so $q, q+1, \cdots$, in $g(j_1, j_2)$ and $h(j_1, j_2)$; and since $j_1 - j_2$ can only take values which make the radicals in $g(j_1, j_2)$ and $h(j_1, j_2)$ real, $j_1 - j_2$ must take: either the values $-q, -q+1, \cdots, -p-2$, in $g(j_1, j_2)$, $-q+1, -q+2, \cdots, -p-1$, in $h(j_1, j_2)$, and $-q, -q+1, \cdots, -p-1$, in $f(j_1, j_2)$ and $k(j_1, j_2)$; or the values $p+1, p+2, \cdots, q-1$, in $g(j_1, j_2)$, $p+2, p+3, \cdots, q$, in $h(j_1, j_2)$, and $p+1, p+2, \cdots, q$, in $f(j_1, j_2)$ and $k(j_1, j_2)$: these results hold even for $q - 1 = p > 0$, when there are no values for $g(j_1, j_2)$ and $h(j_1, j_2)$.

For $q > p = 0$, the above two representations still exist with the expected limiting forms when $q - 1 = p = 0$: the limiting cases of equality in (9.41) above are reducible to the sum of these two representations. In addition a solution is possible with $j_1 - j_2 = 0$ in $f(j_1, j_2)$ and $k(j_1, j_2)$ and no values for

$g(j_1, j_2)$ and $h(j_1, j_2)$; then $j_1 + j_2$ takes the values $q + 1, q + 2, \cdots$, in $f(j_1, j_2)$ and $q, q + 1, \cdots$, in $g(j_1, j_2)$. For $p = 0$, $q = 1$, this gives the limiting case of equality in (9.42) above.

Rearranging these results to correspond to $(p + \frac{1}{2})^2 \nless (q + \frac{1}{2})^2$, and describing the representations by the values which $j_1 + j_2$ and $j_1 - j_2$ take in β, β', we have:—

For

$$-\infty < (q + \tfrac{1}{2})^2 < (\tfrac{1}{2})^2,$$
$$(p + \tfrac{1}{2})^2 = (\tfrac{1}{2})^2, (\tfrac{3}{2})^2, (\tfrac{5}{2})^2, \cdots,$$

one representation each with

$$j_1 + j_2 = p, p + 1, p + 2, \cdots,$$
$$j_1 - j_2 = -p, -p + 1, \cdots, p.$$

For

$$(q + \tfrac{1}{2})^2 = (\tfrac{1}{2})^2,$$
$$(\tfrac{1}{2})^2 \ngtr (p + \tfrac{1}{2})^2 < (\tfrac{3}{2})^2,$$

one representation each with

$$j_1 + j_2 = 0, 1, 2, \cdots,$$
$$j_1 - j_2 = 0.$$

For

$$(q + \tfrac{1}{2})^2 = (\tfrac{1}{2})^2,$$
$$(p + \tfrac{1}{2})^2 = (\tfrac{3}{2})^2, (\tfrac{5}{2})^2, (\tfrac{7}{2})^2, \cdots,$$

three representations each with

$$j_1 + j_2 = p, p + 1, p + 2, \cdots,$$

and with

$$j_1 - j_2 = -p, -p + 1, \cdots, -1,$$

or

$$j_1 - j_2 = 0,$$

or

$$j_1 - j_2 = 1, 2, \cdots, p.$$

For

$$(q + \tfrac{1}{2})^2 = (\tfrac{3}{2})^2, (\tfrac{5}{2})^2, (\tfrac{7}{2})^2, \cdots,$$
$$(p + \tfrac{1}{2})^2 = (\tfrac{5}{2})^2, (\tfrac{7}{2})^2, (\tfrac{9}{2})^2, \cdots, > (q + \tfrac{1}{2})^2,$$

two representations each with

$$j_1 + j_2 = p, p + 1, p + 2, \cdots,$$

and with

$$j_1 - j_2 = -p, -p + 1, \cdots, -q - 1,$$

or

$$j_1 - j_2 = q + 1, q + 2, \cdots, p.$$

II. When only half odd integral values of $j_1 + j_2$ and therefore also of $j_1 - j_2$ are involved, the same arguments as in (I) apply, but there are no special cases, and we have:—

For

$$-\infty < (q + \tfrac{1}{2})^2 \ngtr 0,$$
$$(p + \tfrac{1}{2})^2 = 1^2, 2^2, 3^2, \cdots,$$

one representation each with

$$j_1 + j_2 = p, p + 1, p + 2, \cdots,$$
$$j_1 - j_2 = -p, -p + 1, \cdots, p.$$

For

$$(q + \tfrac{1}{2})^2 = 1^2, 2^2, 3^2, \cdots,$$
$$(p + \tfrac{1}{2})^2 = 2^2, 3^2, 4^2, \cdots, > (q + \tfrac{1}{2})^2,$$

two representations each with

$$j_1 + j_2 = p, p + 1, p + 2, \cdots$$

and with

$$j_1 - j_2 = -p, -p + 1, \cdots, -q - 1,$$

or

$$j_1 - j_2 = q + 1, q + 2, \cdots, p.$$

None of these representations are matrices of finite order. If it is not required that the representation be Hermitian, there are many further possibilities, including, both for integral values of p and q and for half odd integral values, finite sets where $j_1 + j_2$ takes the values $q, q + 1, \cdots, p - 1$, and $j_1 - j_2$ takes the values $-q, -q + 1, \cdots, q$; which are indeed complex transformations of the Hermitian representations of the operators of the rotation group in five dimensions, from which the algebraic form of (7.41) and (7.42) could have been derived.

Case (I) above corresponds to one-valued, case (II) to two-valued representations in the De Sitter space.

Ohio State University
Columbus, Ohio.

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DER DREIERSTOSS

VON CARL LUDWIG SIEGEL

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EINLEITUNG

Es seien A_1, A_2, A_3 drei Massenpunkte im Raum, die sich nach dem Newtonschen Gravitationsgesetz anziehen. Es seien x_k, y_k, z_k die rechtwinkligen kartesischen Koordinaten des Punktes A_k ($k = 1, 2, 3$) und m_k seine Masse; ferner seien die Abstände A_2A_3, A_3A_1, A_1A_2 mit r_1, r_2, r_3 bezeichnet. Setzt man

$$U = \frac{m_2 m_3}{r_1} + \frac{m_3 m_1}{r_2} + \frac{m_1 m_2}{r_3} \tag{1}$$

und versteht unter w_k eine der drei Koordinaten x_k, y_k, z_k, so lauten die Differentialgleichungen des Dreikörperproblems

$$m_k \frac{d^2 w_k}{dt^2} = \frac{\partial U}{\partial w_k} \qquad (w_k = x_k, y_k, z_k;\ k = 1, 2, 3). \tag{2}$$

Man gebe zur Zeit $t = t_0$ irgend welche endlichen Anfangswerte der 9 Koordinaten w_k, für welche die Abstände r_1, r_2, r_3 sämtlich grösser als 0 sind, und ausserdem beliebige endliche Anfangswerte der 9 Geschwindigkeitskomponenten $\frac{dw_k}{dt}$. Nach einem bekannten Existenzsatz aus der Theorie der Differentialgleichungen sind dann die durch jene Anfangswerte bestimmten Lösungen des Systemes (2) in der Umgebung von $t = t_0$ reguläre analytische Funktionen von t. Wir denken uns die Lösungen längs der reellen Achse von $t = t_0$ aus nach beiden Seiten analytisch fortgesetzt und nehmen an, dass wir auf diese Weise zu einem singulären Punkt $t = t_1$ einer der Funktionen w_k gelangen. Da die Differentialgleichungen (2) in sich übergehen, wenn zu t eine beliebige Konstante addiert wird und wenn t durch $-t$ ersetzt wird, so kann man $t_1 = 0$, $t_0 > 0$ voraussetzen.

Wie Sundman[1] gezeigt hat, bestehen bei dem Grenzübergang $t \to 0$, $t > 0$ nur die folgenden beiden Möglichkeiten: Entweder strebt genau eine der drei positiven Zahlen r_1, r_2, r_3 gegen 0, während die beiden andern einen positiven Grenzwert haben, oder aber sie streben alle drei gegen 0. Wir wollen diese beiden Fälle weiterhin als Zweierstoss und Dreierstoss bezeichnen. Der Zweierstoss wurde zuerst für das restringierte Dreikörperproblem von Levi-Cività[2] und

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[1] K. F. Sundman, *Recherches sur le problème des trois corps*, Acta Societatis Scientiarum Fennicae, Bd. 34 (1907), Nr. 6.

[2] T. Levi-Cività, *Traiettorie singolari ed urti nel problema ristretto dei tre corpi*, Annali di matematica pura ed applicata, Ser. III[a], Bd. 9 (1904), S. 1–32.

dann allgemein von Bisconcini[3] behandelt. Sundman führte diese Untersuchungen weiter und zeigte insbesondere, dass beim Zweierstoss der Punkt $t = 0$ ein algebraischer Verzweigungspunkt zweiter Ordnung für die Lösungen ist und dass die Koordinaten w_k in der Umgebung dieses Punktes reguläre Funktionen der Ortsuniformisierenden $t^{\frac{1}{3}}$ sind.

Auch der Dreierstoss wurde von Sundman eingehend untersucht. Er bewies, dass beim Dreierstoss die Ausdrücke $r_k t^{-2/3}$ $(k = 1, 2, 3)$ für $t \to 0$ positive Grenzwerte $\hat{r}_k$ haben, für welche entweder $\hat{r}_1 = \hat{r}_2 = \hat{r}_3$ oder $\hat{r}_2 = \hat{r}_3 + \hat{r}_1$ oder $\hat{r}_3 = \hat{r}_1 + \hat{r}_2$ oder $\hat{r}_1 = \hat{r}_2 + \hat{r}_3$ gilt. Da die drei letzten dieser 4 Fälle durch zyklische Vertauschung der Indizes 1, 2, 3 ineinander übergeführt werden können, so wollen wir uns im folgenden nur noch mit dem ersten und zweiten beschäftigen. Zeichnet man ein Dreieck aus den Seiten $\hat{r}_1$, $\hat{r}_2$, $\hat{r}_3$, so erhält man im ersten Fall ein gleichseitiges Dreieck, im zweiten Fall drei Punkte einer Geraden; wir wollen daher von dem gleichseitigen und dem geradlinigen Fall sprechen. Ferner zeigte Sundman, dass bei jeder Dreierstossbahn die Ebene des Dreiecks $A_1A_2A_3$ für variables t nur eine Parallelverschiebung mit konstanter Geschwindigkeit erleidet. Da bekanntlich die Differentialgleichungen des Dreikörperproblems ungeändert bleiben, wenn man ein beliebiges neues rechtwinkliges Koordinatensystem einführt, das einer Translation mit konstanter Geschwindigkeit oder einer Drehung um konstante Winkel unterworfen wird, so braucht man für die weitere Untersuchung des Dreierstosses nur noch das ebene Dreikörperproblem zu behandeln und kann $z_k = 0$ $(k = 1, 2, 3)$ voraussetzen.

Aus den Resultaten Sundmans folgt sofort, dass beim Dreierstoss die drei Winkel des Dreiecks $A_1A_2A_3$ für $t \to 0$ Grenzwerte haben; im gleichseitigen Fall streben sie nämlich sämtlich gegen $\frac{1}{3}\pi$ und im geradlinigen Fall strebt der Winkel bei A_2 gegen π, während die beiden anderen Winkel gegen 0 streben. Es war aber bisher nicht bekannt, ob auch die Winkel der Dreiecksseiten mit den Koordinatenachsen für $t \to 0$ Grenzwerte haben, d. h. ob die drei Körper in bestimmten Richtungen zusammenstossen. Diese Frage wird im folgenden bejahend beantwortet werden. In engem Zusammenhang mit der Entscheidung dieser Frage steht das Problem der Entwickelbarkeit der Koordinaten der kollidierenden Massenpunkte in irreguläre Potenzreihen der Variabeln t. Wir werden solche Reihenentwicklungen wirklich aufstellen und damit zugleich beweisen, dass beim Dreierstoss der Punkt $t = 0$ im allgemeinen ein logarithmischer Verzweigungspunkt für die Lösungen ist. Dieses Ergebnis bezeichnet einen wesentlichen Unterschied gegenüber dem Zweierstoss. Die Reihenentwicklungen werden uns endlich einen vollen Überblick über sämtliche Dreierstossbahnen in der Nähe von $t = 0$ verschaffen. Sieht man zwei Lösungen nicht als verschieden an, wenn sie durch eine Drehung des Koordinatensystems um konstante Winkel oder durch eine Translation mit konstanter Geschwindigkeit ineinander übergeführt werden können, so zeigt es sich, dass in der Nähe

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[3] G. Bisconcini, *Sur le problème des trois corps*, Acta Mathematica, Bd. 30 (1906), S. 49–92.

von $t = 0$ die sämtlichen Dreierstossbahnen im gleichseitigen Fall von 3 Parametern analytisch abhängen und im geradlinigen Fall von 2 Parametern.

Wir wollen das weiterhin abzuleitende hauptsächliche Resultat unserer Untersuchung noch präzis formulieren. Zur Abkürzung werde gesetzt

$$a = \frac{m_2 m_3 + m_3 m_1 + m_1 m_2}{(m_1 + m_2 + m_3)^2}$$

und

$$b = \frac{m_1\{1 + (1-\omega)^{-1} + (1-\omega)^{-2}\} + m_3(1 + \omega^{-1} + \omega^{-2})}{m_1 + m_2\{\omega^{-2} + (1-\omega)^{-2}\} + m_3},$$

wo ω die im Intervall $0 < \omega < 1$ gelegene Lösung der Gleichung

$$m_1\{(1-\omega)^{-2} - (1-\omega)\} + m_2\{\omega(1-\omega)^{-2} - (1-\omega)\omega^{-2}\} + m_3(\omega - \omega^{-2}) = 0$$

bedeutet; ferner sei

$$a_1 = \tfrac{1}{6}\{-1 + [13 + 12\,(1-3a)^{\frac{1}{2}}]^{\frac{1}{2}}\},\ a_2 = \tfrac{1}{6}\{-1 + [13 - 12\,(1-3a)^{\frac{1}{2}}]^{\frac{1}{2}}\},$$

$$b_1 = \tfrac{1}{6}\{-1 + (25 + 16b)^{\frac{1}{2}}\}.$$

Ist dann weder $\dfrac{2}{3a_2}$ noch $\dfrac{a_1}{a_2}$ eine ganze Zahl, so lassen sich sämtliche Koordinaten x_k, y_k $(k = 1, 2, 3)$ im gleichseitigen Falle des Dreierstosses in der Form

$$x_k = t^{2/3} x_k^*(u_1, u_2, u_3), \qquad y_k = t^{2/3} y_k^*(u_1, u_2, u_3) \qquad (k = 1, 2, 3)$$

ausdrücken, wo $x_k^*(u_1, u_2, u_3)$ und $y_k^*(u_1, u_2, u_3)$ Potenzreihen in den Grössen

$$u_1 = \alpha_1 t^{2/3}, \qquad u_2 = \alpha_2 t^{a_1}, \qquad u_3 = \alpha_3 t^{a_2}$$

mit konstanten α_1, α_2, α_3 bedeuten; dabei hängen die Koeffizienten dieser Potenzreihen nur von m_1, m_2, m_3 ab. Die Werte α_1, α_2, α_3 sind eindeutig durch die Dreierstossbahn bestimmt, und umgekehrt liefert jedes System von reellen Werten α_1, α_2, α_3 wieder eine Dreierstossbahn, für welche der gleichseitige Fall vorliegt. Ist ferner $\dfrac{3b_1}{2}$ keine ganze Zahl, so haben die Koordinaten im geradlinigen Fall des Dreierstosses die Form

$$x_k = t^{2/3} x_k^*(v_1, v_2), \qquad y_k = t^{2/3} y_k^*(v_1, v_2) \qquad (k = 1, 2, 3),$$

wo $x_k^*(v_1, v_2)$ und $y_k^*(v_1, v_2)$ Potenzreihen in

$$v_1 = \beta_1 t^{2/3}, \qquad v_2 = \beta_2 t^{b_1}$$

mit konstanten durch die Bahn eindeutig bestimmten Werten β_1, β_2 bedeuten; umgekehrt ergibt jedes reelle System β_1, β_2 eine Dreierstossbahn für den geradlinigen Fall. Ist $\dfrac{2}{3a_2}$ eine ganze Zahl g, so bleiben die obigen Aussagen bestehen, wenn man darin u_1 durch die Formel

$$u_1 = t^{2/3}(\alpha_1 + c_1 \alpha_3^g \log t)$$

erklärt, wo c_1 eine gewisse nur von m_1, m_2, m_3 abhängige Konstante bedeutet. Im Falle eines ganzzahligen $\frac{a_1}{a_2} = h$ hat man entsprechend

$$u_2 = t^{a_1}(\alpha_2 + c_2\alpha_3^h \log t)$$

zu erklären, und für ganzzahliges $\frac{3b_1}{2} = j$ ist

$$v_2 = t^{b_1}(\beta_2 + c_3\beta_1^j \log t)$$

zu setzen, wobei c_2 und c_3 wieder nur von m_1, m_2, m_3 abhängen.

Es sei noch daran erinnert, dass wir im geradlinigen Falle zwei weitere von zwei Parametern abhängige Scharen von Dreierstossbahnen erhalten, wenn wir m_1, m_2, m_3 zyklisch vertauschen. Ausserdem kann man noch eine beliebige Drehung des Koordinatensystems um konstante Winkel vornehmen, sowie eine beliebige Translation mit konstanter Geschwindigkeit.

In den ersten 6 Paragraphen der vorliegenden Arbeit werden im wesentlichen die Ergebnisse, welche Sundman für den Dreierstoss erhalten hat, in etwas veränderter Art hergeleitet. In §7 werden spezielle Dreierstossbahnen, die in den bekannten Lagrangeschen partikulären Lösungen des Dreikörperproblems enthalten sind, kurz besprochen. Als weitere Vorbereitung wird in §8 unter Anwendung der Jacobi-Hamiltonschen Theorie die Transformation der Differentialgleichungen des Dreikörperproblems behandelt, welche man in der Astronomie als Elimination der Knoten bezeichnet.[4] Es wird dann die Bestimmung aller Dreierstossbahnen zurückgeführt auf die Lösung folgender Aufgabe:

Es sei

$$\frac{d\delta_k}{ds} = f_k \qquad (k = 1, \cdots, n) \tag{3}$$

ein System von n Differentialgleichungen erster Ordnung, dessen rechte Seiten Potenzreihen der unbekannten Funktionen $\delta_1, \cdots, \delta_n$ sind, aber nicht die unabhängige Variable s explizit enthalten. Die Reihen f_k mögen keine konstanten Glieder haben, und es sei a_{kl} der Koeffizient von δ_l $(l = 1, \cdots, n)$ in den linearen Gliedern von f_k $(k = 1, \cdots, n)$. Von den charakteristischen Wurzeln der Matrix (a_{kl}) mögen genau p einen negativen Realteil haben und keine den Realteil 0. Man gebe sämtliche Lösungen des Systemes (3) an, welche für $s \to \infty$ den Grenzwert 0 haben.

Diese Aufgabe ist unter noch allgemeineren Voraussetzungen über die Funktionen f_k von Bohl[5] bearbeitet worden. Er zeigte auf sehr geistreiche Weise, dass jene Lösungen von p Parametern abhängen. Die von Bohl verwendeten topologischen Hilfssätze beruhen aber wesentlich auf indirekten Schlüssen und können nicht ohne weiteres zu einer Konstruktion der Lösungen benutzt werden.

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[4] Vergl. S. 339–341 in E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies, with an introduction to the problem of three bodies*, Cambridge (1904).

[5] P. Bohl, *Sur certaines équations différentielles d'un type général utilisables en mécanique*, Bulletin de la société mathématique de France, Bd. 38 (1910), S. 5–138.

Deshalb wird im folgenden in den Paragraphen 11 und 12 das Problem auf eine andere Art behandelt werden, durch Benutzung gewisser Reihentransformationen, welche die gesuchten Lösungen in expliziter Gestalt ergeben. Die irregulären Potenzreihen, durch welche wir die Lösungen ausdrücken werden, treten bereits in Poincaré's Untersuchungen über asymptotische Bahnkurven auf; dort fehlt aber gerade der Nachweis, dass diese Reihen wirklich alle asymptotischen Bahnen darstellen. Die Art, in der wir die Cauchysche Majorantenmethode anwenden, weicht von der üblichen ein wenig ab und bietet vielleicht auch ein selbständiges Interesse, da sie fast ohne Rechnung zum Ziele führt.

Im letzten Paragraphen werden schliesslich unsere Resultate über das System (3) angewendet auf die geeignet transformierten Differentialgleichungen des Dreikörperproblemes und ergeben den oben ausgesprochenen Satz über die Mannigfaltigkeit der Dreierstossbahnen.

1. Algebraische Vorbereitungen

Ist F eine Funktion der Zeit t, so soll die erste und die zweite Ableitung von F nach t in üblicher Weise mit $\dot{F}$ und $\ddot{F}$ bezeichnet werden. Unter $\gamma_1, \cdots, \gamma_{17}$ wollen wir weiterhin Grössen verstehen, die längs der zu betrachtenden Bahnkurve des Dreikörperproblems konstant sind. Zur Abkürzung werde noch

$$m = m_1 + m_2 + m_3 \tag{4}$$

gesetzt.

Wir schreiben zunächst die bekannten algebraischen Integrale des Dreikörperproblems auf, nämlich die Flächenintegrale, die Schwerpunktsintegrale und das Energieintegral. Die Flächenintegrale lauten

$$\begin{gathered}\sum_{k=1}^{3} m_k(y_k\dot{z}_k - z_k\dot{y}_k) = \gamma_1, \qquad \sum_{k=1}^{3} m_k(z_k\dot{x}_k - x_k\dot{z}_k) = \gamma_2,\\ \sum_{k=1}^{3} m_k(x_k\dot{y}_k - y_k\dot{x}_k) = \gamma_3.\end{gathered}\tag{5}$$

Die Schwerpunktsintegrale besagen, dass der Schwerpunkt der drei Massenpunkte sich geradlinig und gleichförmig bewegt, nämlich

$$\frac{1}{m}\sum_{k=1}^{3} m_k x_k = \gamma_4 t + \gamma_5, \qquad \frac{1}{m}\sum_{k=1}^{3} m_k y_k = \gamma_6 t + \gamma_7,$$

$$\frac{1}{m}\sum_{k=1}^{3} m_k z_k = \gamma_8 t + \gamma_9.$$

Da die Differentialgleichungen (2) sich nicht ändern, wenn x_k, y_k, z_k durch $x_k + \gamma_4 t + \gamma_5$, $y_k + \gamma_6 t + \gamma_7$, $z_k + \gamma_8 t + \gamma_9$ $(k = 1, 2, 3)$ ersetzt werden, so kann man weiterhin voraussetzen, dass der Schwerpunkt fest im Koordinatenanfangspunkt liegt, dass also die Gleichungen

$$\sum_{k=1}^{3} m_k x_k = 0, \qquad \sum_{k=1}^{3} m_k y_k = 0, \qquad \sum_{k=1}^{3} m_k z_k = 0 \tag{6}$$

gelten. Bedeutet

$$T = \tfrac{1}{2} \sum_{k=1}^{3} m_k(\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2) \tag{7}$$

die lebendige Kraft des Punktsystems, so ist

$$T - U = \gamma_{10} \tag{8}$$

das Energieintegral.

Wir setzen nun

$$J = \sum_{k=1}^{3} m_k(x_k^2 + y_k^2 + z_k^2). \tag{9}$$

Dann ist

$$\tfrac{1}{2}\dot{J} = \sum_{k=1}^{3} m_k(x_k\dot{x}_k + y_k\dot{y}_k + z_k\dot{z}_k), \tag{10}$$

$$\tfrac{1}{2}\ddot{J} = \sum_{k=1}^{3} m_k(\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2) + \sum_{k=1}^{3} m_k(x_k\ddot{x}_k + y_k\ddot{y}_k + z_k\ddot{z}_k),$$

also nach (2) und (7)

$$\tfrac{1}{2}\ddot{J} = 2T + \sum_{k=1}^{3}\left(x_k\frac{\partial U}{\partial x_k} + y_k\frac{\partial U}{\partial y_k} + z_k\frac{\partial U}{\partial z_k}\right). \tag{11}$$

Da nun U eine homogene Funktion der Dimension -1 in den 9 Variabeln $x_1, \cdots, z_3$ ist, so ist nach einem bekannten Eulerschen Satze

$$\sum_{k=1}^{3}\left(x_k\frac{\partial U}{\partial x_k} + y_k\frac{\partial U}{\partial y_k} + z_k\frac{\partial U}{\partial z_k}\right) = -U$$

und (11) ergibt die Lagrangesche Formel

$$\tfrac{1}{2}\ddot{J} = 2T - U,$$

also nach (8)

$$\tfrac{1}{2}\ddot{J} = T + \gamma_{10}, \tag{12}$$

$$\tfrac{1}{2}\ddot{J} = U + 2\gamma_{10}. \tag{13}$$

Sind $B_1, \cdots, B_n$ und $C_1, \cdots, C_n$ zwei Reihen von je n Grössen und bezeichnet man mit

$$D_{pq} = B_pC_q - B_qC_p \qquad (1 \leqq p < q \leqq n)$$

die $n(n-1)/2$ zweireihigen Unterdeterminanten der aus diesen beiden Reihen gebildeten Matrix, so gilt identisch

$$\sum_{p=1}^{n} B_p^2 \sum_{p=1}^{n} C_p^2 - \left(\sum_{p=1}^{n} B_pC_p\right)^2 = \sum_{p,q} D_{pq}^2. \tag{14}$$

Wir wählen $n = 9$ und identifizieren die Paare B_p, C_p $(p = 1, \cdots, 9)$ mit $w_k\sqrt{m_k}$, $\dot{w}_k\sqrt{m_k}$ $(w_k = x_k, y_k, z_k; k = 1, 2, 3)$. Behält man die Abkürzung

D_{pq} bei, so geht (14) zufolge (7), (9), (10) über in

$$2TJ - \tfrac{1}{4}\dot{J}^2 = \sum_{p,q} D_{pq}^2. \tag{15}$$

Also gilt insbesondere die Ungleichung

$$\dot{J}^2 \leqq 8JT. \tag{16}$$

2. Die Singularitäten der Bahnkurve

Wir wenden auf die Differentialgleichungen des Dreikörperproblems folgenden bekannten Existenzsatz an:

Es seien $F_1, \cdots, F_n$ Funktionen von n Variabeln $s_1, \cdots, s_n$, die in der Umgebung eines Punktes $s_1 = \sigma_1, \cdots, s_n = \sigma_n$ in Reihen nach Potenzen von $s_1 - \sigma_1, \cdots, s_n - \sigma_n$ entwickelbar sind. Es sei δ eine positive Zahl, für welche die Potenzreihen in dem Gebiete

$$|s_1 - \sigma_1| \leqq \delta, \cdots, |s_n - \sigma_n| \leqq \delta \tag{17}$$

konvergieren; ferner sei K eine gemeinsame obere Schranke der Werte $|F_1|, \cdots, |F_n|$ in diesem Gebiete. Es sei t eine reelle Variable und t_0 irgend ein Wert von t. Es gibt genau ein System von n Funktionen $\varphi_1, \cdots, \varphi_n$ der reellen Variabeln t mit folgenden Eigenschaften:

1) Die Funktion $\varphi_k(t)$ ist differentiierbar in einer Umgebung von $t = t_0$ und es ist $\varphi_k(t_0) = \sigma_k$ $(k = 1, \cdots, n)$;

2) in dieser Umgebung von $t = t_0$ genügen $s_1 = \varphi_1(t), \cdots, s_n = \varphi_n(t)$ den Differentialgleichungen

$$\frac{ds_k}{dt} = F_k \qquad (k = 1, \cdots, n).$$

Es gibt ausserdem eine nur von δ, K und n abhängige positive Zahl τ, sodass die Funktionen $s_k = \varphi_k(t)$ in Reihen nach Potenzen von $t - t_0$ entwickelbar sind, welche sämtlich für $|t - t_0| < \tau$ konvergieren und in diesem Gebiete den Ungleichungen (17) genügen.

Um diesen Satz auf das System (2) anzuwenden, wählen wir $n = 18$ und setzen für $k = 1, 2, 3$

$$x_k = s_k, \qquad y_k = s_{k+3}, \qquad z_k = s_{k+6},$$

$$\dot{x}_k = s_{k+9} = F_k, \qquad \dot{y}_k = s_{k+12} = F_{k+3}, \qquad \dot{z}_k = s_{k+15} = F_{k+6},$$

$$\frac{1}{m_k}\frac{\partial U}{\partial x_k} = F_{k+9}, \qquad \frac{1}{m_k}\frac{\partial U}{\partial y_k} = F_{k+12}, \qquad \frac{1}{m_k}\frac{\partial U}{\partial z_k} = F_{k+15}.$$

Zur Zeit $t = t_0$ seien irgend welche endlichen reellen Anfangswerte $\sigma_1, \cdots, \sigma_{18}$ der 18 Grössen $x_1, \cdots, \dot{z}_3$ gegeben, für welche keine Kollision vorliegt, also die Abstände r_1, r_2, r_3 grösser als 0 sind. Wir berechnen aus den Zahlen $\sigma_1, \cdots, \sigma_{18}$ nach (1) und (8) den Anfangswert $U = U_0$ und den Wert der Konstanten γ_{10} des Energieintegrales. Ist dann K_1 irgend eine obere Schranke für

U_0, so gelten nach (1), (7), (8) für die Anfangswerte die Ungleichungen

$$r_1^{-1} < \frac{K_1}{m_2 m_3}, \qquad r_2^{-1} < \frac{K_1}{m_3 m_1}, \qquad r_3^{-1} < \frac{K_1}{m_1 m_2},$$

$$\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2 \leqq 2(K_1 + \gamma_{10})m_k^{-1} \qquad (k = 1, 2, 3).$$

Folglich lassen sich zwei positive Grössen δ und K als Funktionen von m_1, m_2, m_3, K_1 und γ_{10} allein so wählen, dass die Voraussetzungen des Satzes erfüllt sind. Die durch die Anfangswerte eindeutig bestimmte Lösung $s_k = \varphi_k(t)$ $(k = 1, \cdots, 18)$ der Differentialgleichungen ist dann regulär in einem Zeitintervall $t_0 - \tau < t < t_0 + \tau$, wo die positive Zahl τ nur von m_1, m_2, m_3, K_1, γ_{10} abhängt.

Wir betrachten jetzt, von $t = t_0$ ausgehend, die Funktionen $\varphi_k(t)$ für fallende reelle Werte von t. Entweder sind sie sämtlich regulär für alle endlichen reellen $t \leqq t_0$ oder aber es gibt einen endlichen Wert $t = t_1 < t_0$, sodass alle Funktionen im links offenen Intervall $t_1 < t \leqq t_0$ regulär sind und mindestens eine von ihnen für $t = t_1$ singulär ist. Eine analoge Aussage gilt für $t \geqq t_0$ und wachsende Werte von t. Da die Differentialgleichungen (2) bei der Transformation $t \to -t$ in sich übergehen, so kann man sich auf die Untersuchung für fallendes $t \leqq t_0$ beschränken. Weil die Differentialgleichungen die Zeit t nicht explizit enthalten, so kann man ausserdem noch $t_1 = 0$ voraussetzen. Wegen der Regularität für $t_0 - \tau < t < t_0 + \tau$ ist dann $\tau \leqq t_0$. Nun sei $0 < t_2 < \tau$ und $U = U_2$ der Wert von U für $t = t_2$. Wäre auch $U_2 < K_1$, so wären die Funktionen $\varphi_k(t)$ auch sämtlich in dem Intervall $t_2 - \tau < t < t_2 + \tau$ regulär; aber dies ist ein Widerspruch, da das Intervall den singulären Punkt $t = 0$ enthält. Für $0 < t < \tau$ gilt daher $U \geqq K_1$, und dabei hängt τ nur von m_1, m_2, m_3, γ_{10} und K_1 ab. Da K_1 beliebig gross sein kann, so folgt, dass U über alle Schranken wächst, wenn t zu 0 abnimmt. Dies bedeutet, dass für $t \to 0$ der kleinste der 3 Abstände r_1, r_2, r_3 gegen 0 strebt.

In folgenden bedeuten τ_1, τ_2, τ_3 geeignete hinreichend klein zu wählende positive Zahlen, die nur von den Massen m_1, m_2, m_3 und den gegebenen Anfangswerten von $x_1, \cdots, \dot{z}_3$ abhängen. Auf der betrachteten Bahnkurve ist $U \to \infty$ für $t \to 0$, also nach (13)

$$\ddot{J} > 0 \qquad (0 < t \leqq \tau_1). \tag{18}$$

Daher ist J im Intervall $0 < t \leqq \tau_1$ eine konvexe Funktion von t und hat folglich für $t \to 0$ einen Grenzwert J_0, der positiv oder 0 sein kann. Im Falle $J_0 > 0$ folgt aus (6) und (9), dass der grösste der drei Abstände r_1, r_2, r_3 für $t \to 0$ oberhalb einer positiven Schranke bleibt. Da andererseits der kleinste dieser Abstände gegen 0 strebt und sie sämtlich für $t > 0$ stetige Funktionen von t sind, so ergibt sich, dass für $t \to 0$ eine bestimmte der Dreiecksseiten r_k gegen 0 strebt, während die beiden anderen oberhalb einer positiven Schranke bleiben; es stossen dann also zur Zeit $t = 0$ genau zwei von den Körpern zusammen. Dieser Zweierstoss ist von Sundman vollständig untersucht worden. Er hat gezeigt,

dass die Koordinaten der drei Körper in der Umgebung von $t = 0$ reguläre Funktionen der Ortsuniformisierenden $t^{1/3}$ sind.

Weiterhin beschäftigen wir uns dauernd mit dem Fall $J_0 = 0$. Dann streben aber nach (9) für $t \to 0$ alle drei Abstände r_1, r_2, r_3 gegen 0 und es stossen zur Zeit $t = 0$ alle drei Körper im Nullpunkt zusammen. Es soll genauer untersucht werden, in welcher Weise dieser Dreierstoss vor sich geht.

3. Das asymptotische Verhalten von J und $\dot{J}$

Nach (18) ist die positive Funktion J konvex im Intervall $0 < t \leqq \tau_1$, ferner strebt J gegen 0 für $t \to 0$. Folglich gilt

$$\dot{J} > 0 \qquad (0 < t \leqq \tau_1). \tag{19}$$

Andererseits ist nach (12)

$$\ddot{J}J^{-1/4} - \tfrac{1}{4}\dot{J}^2J^{-5/4} = \tfrac{1}{4}(8JT - \dot{J}^2)J^{-5/4} + 2\gamma_{10}J^{-1/4},$$

und hierin ist die linke Seite gerade die Ableitung der Funktion $\dot{J}J^{-1/4}$ nach t. Integriert man diese Gleichung zwischen den Grenzen t und τ_1 und bezeichnet mit J_1 und $\dot{J}_1$ die Werte von J und $\dot{J}$ für $t = \tau_1$, so folgt

$$\dot{J}_1J_1^{-\frac{1}{4}} - \dot{J}J^{-\frac{1}{4}} = \tfrac{1}{4}\int_t^{\tau_1}(8JT - \dot{J}^2)J^{-5/4}\,dt + 2\gamma_{10}\int_t^{\tau_1}J^{-\frac{1}{4}}\,dt. \tag{20}$$

Wir wollen nun beweisen, dass die beiden Integrale auf der rechten Seite für $t \to 0$ endliche Grenzwerte haben. Wir verstehen weiterhin unter $\mu_1, \cdots, \mu_5$ gewisse positive Zahlen, die genügend klein zu wählen sind und nur von m_1, m_2, m_3 abhängen. Da der Nullpunkt im Schwerpunkt des Dreiecks $A_1A_2A_3$ liegt, so ist

$$J > \mu_1(r_1^2 + r_2^2 + r_3^2)$$

und folglich

$$U > \mu_2J^{-\frac{1}{2}},$$

also auch

$$U + 2\gamma_{10} > \mu_3J^{-\frac{1}{2}} \qquad (0 < t < \tau_2).$$

Im Intervall $0 < t < \tau_2$ gilt dann nach (13) und (19)

$$(\dot{J}^2)^{\cdot} = 2\dot{J}\ddot{J} > 4\mu_3\dot{J}J^{-\frac{1}{2}}$$

$$\dot{J}^2 > 8\mu_3J^{\frac{1}{2}}$$

$$\dot{J}J^{-1/4} > \mu_4 \tag{21}$$

$$J > \mu_5t^{4/3}.$$

Demnach konvergiert das zweite Integral in (20) bis nach $t = 0$. Zufolge (19) und (20) ist dann das erste Integral in (20) für $t \to 0$ nach oben beschränkt; da

aber sein Integrand wegen (16) nicht-negativ ist, so konvergiert es ebenfalls bis nach $t = 0$. Nach (20) existiert also

(22) $$\lim_{t\to 0} \dot{J}J^{-\frac{1}{4}} = \gamma_{11},$$

und zwar ist dieser Grenzwert zufolge (21) eine positive Zahl. Durch Integration von $\dot{J}J^{-1/4}$ folgt aus (22) die asymptotische Gleichung

$$J^{3/4} \sim \tfrac{3}{4}\gamma_{11}t \qquad (t \to 0)$$

(23) $$J \sim \lambda t^{4/3}$$

mit

(24) $$\lambda = (\tfrac{3}{4}\gamma_{11})^{4/3} > 0,$$

also nach (22)

(25) $$\dot{J} \sim \tfrac{4}{3}\lambda t^{1/3}.$$

Damit ist das asymptotische Verhalten von J und $\dot{J}$ klargelegt.

4. Das asymptotische Verhalten von U und T

Setzt man

(26) $$(8JT - \dot{J}^2)t^{-2/3} = g(t),$$

so ist nach (16)

(27) $$g(t) \geqq 0$$

und nach §3 das Integral

$$\int_0^{\tau_1} g(t)t^{2/3}J^{-5/4}\,dt$$

konvergent. Zufolge (23), (24), (25) konvergiert dann auch

(28) $$\int_0^{\tau_1} g(t)\,\frac{dt}{t}.$$

Wir wollen beweisen, dass $g(t)$ für $t \to 0$ den Grenzwert 0 besitzt. Wegen (27) und der Konvergenz des Integrales (28) ist jedenfalls

(29) $$\liminf_{t\to 0} g(t) = 0.$$

Wäre nun

$$\limsup_{t\to 0} g(t) > 0,$$

so könnte man wegen (29) und der Stetigkeit von $g(t)$ für $t > 0$ eine positive Zahl γ_{12} und eine monoton zu 0 abnehmende Folge $\tau_1 > t_1 > t_2 > \cdots$ so finden, dass

$$(30) \qquad g(t_{2n}) = \gamma_{12}, \qquad g(t_{2n-1}) = 3\gamma_{12} \qquad (n = 1, 2, 3, \cdots)$$

und in jedem Intervall $t_{2n} \leqq t \leqq t_{2n-1}$ die Ungleichung

$$(31) \qquad \gamma_{12} \leqq g(t) \leqq 3\gamma_{12}$$

gilt.

Nach (26) ist

$$T = \tfrac{1}{8}\{\dot{J}^2 + g(t)t^{2/3}\}J^{-1},$$

also nach (23), (25), (31)

$$(32) \qquad T < \gamma_{13}t^{-2/3} \qquad (t_{2n} \leqq t \leqq t_{2n-1}\,;\, n = 1, 2, 3, \cdots).$$

Ferner ist

$$\dot{T} = \dot{U} = \sum_{k=1}^{3}\left(\frac{\partial U}{\partial x_k}\dot{x}_k + \frac{\partial U}{\partial y_k}\dot{y}_k + \frac{\partial U}{\partial z_k}\dot{z}_k\right),$$

also nach (1), (7), (8), (32)

$$(33) \qquad |\,\dot{T}\,| < \gamma_{14}U^2T^{\frac{1}{2}} < \gamma_{15}t^{-5/3},$$

wieder im Intervall $t_{2n} \leqq t \leqq t_{2n-1}$ $(n = 1, 2, 3, \cdots)$. Folglich gilt dort auch nach (23), (25), (32), (33) die Ungleichung

$$|\,(8JTt^{-2/3})^{\cdot}\,| < \gamma_{16}t^{-1},$$

und die Funktion $8JTt^{-2/3}$ ändert sich daher im Intervall $t_{2n} \leqq t \leqq t_{2n-1}$ um weniger als

$$\gamma_{16}\int_{t_{2n}}^{t_{2n-1}}\frac{dt}{t}.$$

In dem gleichen Intervall ändert sich ferner die Funktion $\dot{J}^2t^{-2/3}$ zufolge (25) um weniger als γ_{12}, wenn nur $t_{2n-1} < \tau_3$ ist, also für alle genügend grossen Werte von n. Nach (26) und (30) erhält man dann

$$2\gamma_{12} = g(t_{2n-1}) - g(t_{2n}) < \gamma_{12} + \gamma_{16}\int_{t_{2n}}^{t_{2n-1}}\frac{dt}{t},$$

also

$$\int_{t_{2n}}^{t_{2n-1}}\frac{dt}{t} > \frac{\gamma_{12}}{\gamma_{16}}$$

und nach (31)

$$(34) \qquad \int_{t_{2n}}^{t_{2n-1}} g(t)\,\frac{dt}{t} > \gamma_{17} > 0,$$

für alle genügend grossen n. Durch Summation über n folgt aber aus (34) ein Widerspruch gegen die Konvergenz des Integrales (28).

Damit ist bewiesen, dass $g(t)$ für $t \to 0$ den Grenzwert 0 besitzt. Es ist also auch

$$8JT - \dot{J}^2 = o(t^{2/3}) \tag{35}$$

und nach (23), (25)

$$T \sim \tfrac{2}{9}\lambda t^{-2/3}, \tag{36}$$

nach (8)

$$U \sim \tfrac{2}{9}\lambda t^{-2/3}. \tag{37}$$

5. Ebene Bewegung

Aus (15) und (35) ersieht man, dass jede der 36 in (15) auftretenden zweireihigen Determinanten D_{pq} für $t \to 0$ von kleinerer Grössenordnung als $t^{1/3}$ ist. Sind w und w_0 irgend zwei der 9 Koordinaten $x_1, \cdots, z_3$, so gilt also

$$w\dot{w}_0 - w_0\dot{w} = o(t^{1/3}). \tag{38}$$

Nach (5) haben daher die Flächenkonstanten $\gamma_1, \gamma_2, \gamma_3$ alle drei den Wert 0. Hieraus ergibt sich nun leicht, dass die Bewegung in einer festen Ebene durch den Nullpunkt vor sich geht:

Die Differentialgleichungen (2) sind invariant gegenüber einer Drehung des Koordinatensystems um konstante Winkel. Deshalb kann man annehmen, dass zur Zeit $t = t_0$ die drei Punkte A_1, A_2, A_3 in der Ebene $z = 0$ liegen. Wegen $\gamma_1 = 0, \gamma_2 = 0$ ist dann also

$$\sum_{k=1}^{3} m_k y_k \dot{z}_k = 0, \qquad \sum_{k=1}^{3} m_k x_k \dot{z}_k = 0 \tag{39}$$

für $t = t_0$. Nach (6) ist ferner

$$\sum_{k=1}^{3} m_k \dot{z}_k = 0. \tag{40}$$

Liegen nun die drei Massenpunkte zur Zeit $t = t_0$ nicht auf einer Geraden, so ist die aus den drei Zeilen x_1, x_2, x_3; y_1, y_2, y_3; 1, 1, 1 gebildete Determinante von 0 verschieden, und aus (39), (40) folgt das Verschwinden der drei Werte $\dot{z}_1, \dot{z}_2, \dot{z}_3$ für $t = t_0$. Liegen andererseits die drei Punkte zur Zeit $t = t_0$ auf einer Geraden, so kann man durch eine Drehung des Koordinatensystems erreichen, dass diese Gerade die x-Achse wird und ausserdem die Geschwindigkeitskomponente $\dot{z}_3$ für $t = t_0$ verschwindet. Nach (39) und (40) ist dann aber

$$m_1x_1\dot{z}_1 + m_2x_2\dot{z}_2 = 0, \qquad m_1\dot{z}_1 + m_2\dot{z}_2 = 0 \tag{41}$$

für $t = t_0$. Da zur Zeit $t = t_0$ keine Kollision stattfindet, so ist dann $x_1 \neq x_2$, und aus (41) folgt wieder das Verschwinden von $\dot{z}_1$ und $\dot{z}_2$ für $t = t_0$. In jedem Fall liegen also die Richtungen der Bewegungen der drei Massenpunkte zur Zeit $t = t_0$ ebenfalls in der Ebene $z = 0$, und aus den Differentialgleichungen

(2) folgt nach dem Eindeutigkeitssatz, dass die Bewegung ganz in der Ebene $z = 0$ erfolgt.

Wir können daher weiterhin $z_k = 0$ $(k = 1, 2, 3)$ annehmen und brauchen nur noch ebene Bewegungen zu betrachten.

6. Das asymptotische Verhalten der Dreiecksseiten

Bedeutet w eine beliebige der 6 Koordinaten x_k, y_k $(k = 1, 2, 3)$, so wollen wir die Abkürzung

$$w^* = wt^{-2/3} \tag{42}$$

einführen, also z.B. $x_1^* = x_1 t^{-2/3}$. Ist allgemeiner Φ eine homogene Funktion von $x_1, \cdots, y_3$, so wollen wir unter Φ^* den Wert verstehen, den man erhält, wenn man in Φ alle Variabeln w durch w^* ersetzt; z.B. ist also

$$r_k^* = r_k t^{-2/3}, \qquad U^* = U t^{2/3}.$$

Nach (23) und (36) gilt die Abschätzung

$$w = O(t^{2/3}), \qquad \dot{w} = O(t^{-1/3}). \tag{43}$$

Hieraus folgt nach (42)

$$w^* = O(1), \qquad \dot{w}^* = \dot{w}t^{-2/3} - \tfrac{2}{3}wt^{-5/3} = O(t^{-1}). \tag{44}$$

Ferner ist nach (38), wenn w_0 ebenfalls eine Koordinate bedeutet,

$$w^*\dot{w}_0^* - w_0^*\dot{w}^* = wt^{-2/3}(\dot{w}_0 t^{-2/3} - \tfrac{2}{3}w_0 t^{-5/3}) - w_0 t^{-2/3}(\dot{w}t^{-2/3} - \tfrac{2}{3}wt^{-5/3}) = o(t^{-1}) \tag{45}$$

und nach (23), (26)

$$J^* \sim \lambda, \qquad \dot{J}^* = \dot{J}t^{-4/3} - \tfrac{4}{3}Jt^{-7/3} = o(t^{-1}). \tag{46}$$

Aus (44) und (45) erhält man

$$J^*\dot{w}^* - \tfrac{1}{2}w^*\dot{J}^* = \sum_{k=1}^{3} m_k\{x_k^*(x_k^*\dot{w}^* - w^*\dot{x}_k^*) + y_k^*(y_k^*\dot{w}^* - w^*\dot{y}_k^*)\} = o(t^{-1}) \tag{47}$$

und aus (44) und (46)

$$J^*\dot{w}^* - \tfrac{1}{2}w^*\dot{J}^* = \lambda\dot{w}^* + o(t^{-1}). \tag{48}$$

Aus (24), (47), (48) ergibt sich

$$\dot{w}^* = o(t^{-1}). \tag{49}$$

Es ist

$$\frac{\partial U^*}{\partial w^*} = \frac{\partial U}{\partial w} t^{4/3}$$

und folglich gehen die Differentialgleichungen (2) durch die Substitution (42)

Damit ist bewiesen, dass $g(t)$ für $t \to 0$ den Grenzwert 0 besitzt. Es ist also auch

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Wir können daher weiterhin $z_k = 0$ $(k = 1, 2, 3)$ annehmen und brauchen nur noch ebene Bewegungen zu betrachten.

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$$w^*\dot{w}_0^* - w_0^*\dot{w}^* = wt^{-2/3}(\dot{w}_0 t^{-2/3} - \tfrac{2}{3}w_0 t^{-5/3}) - w_0 t^{-2/3}(\dot{w}t^{-2/3} - \tfrac{2}{3}wt^{-5/3}) = o(t^{-1}) \tag{45}$$

und nach (23), (26)

$$J^* \sim \lambda, \qquad \dot{J}^* = \dot{J}t^{-4/3} - \tfrac{4}{3}Jt^{-7/3} = o(t^{-1}). \tag{46}$$

Aus (44) und (45) erhält man

$$J^*\dot{w}^* - \tfrac{1}{2}w^*\dot{J}^* = \sum_{k=1}^{3} m_k\{x_k^*(x_k^*\dot{w}^* - w^*\dot{x}_k^*) + y_k^*(y_k^*\dot{w}^* - w^*\dot{y}_k^*)\} = o(t^{-1}) \tag{47}$$

und aus (44) und (46)

$$J^*\dot{w}^* - \tfrac{1}{2}w^*\dot{J}^* = \lambda\dot{w}^* + o(t^{-1}). \tag{48}$$

Aus (24), (47), (48) ergibt sich

$$\dot{w}^* = o(t^{-1}). \tag{49}$$

Es ist

$$\frac{\partial U^*}{\partial w^*} = \frac{\partial U}{\partial w}\,t^{4/3}$$

und folglich gehen die Differentialgleichungen (2) durch die Substitution (42)

über in

$$m_k(w_k^* t^{2/3})^{\cdot\cdot} = \frac{\partial U^*}{\partial w_k^*} t^{-4/3} \qquad (w_k = x_k, y_k; k = 1, 2, 3)$$

$$-\tfrac{2}{9} w_k^* + (\dot{w}_k^* t^{4/3})^{\cdot} t^{2/3} = \frac{1}{m_k} \frac{\partial U^*}{\partial w_k^*}. \tag{50}$$

Nun sei $\epsilon > 0$, $\epsilon \to 0$. Im Intervall $\epsilon \leqq t \leqq 2\epsilon$ gilt nach (49)

$$w^* = (w^*)_{t=\epsilon} + o\left(\int_\epsilon^{2\epsilon} \frac{dt}{t}\right) = (w^*)_{t=\epsilon} + o(1). \tag{51}$$

Ferner ist nach (37)

$$U^* \sim \tfrac{2}{9}\lambda;$$

also sind für $t \to 0$ die reziproken Werte der Grössen r_k^* $(k = 1, 2, 3)$ beschränkt und das gleiche gilt dann für die zweiten partiellen Ableitungen von U^* nach seinen Variabeln w^*. Nach dem Mittelwertsatz folgt aus (51) demnach

$$\frac{\partial U^*}{\partial w^*} = \left(\frac{\partial U^*}{\partial w^*}\right)_{t=\epsilon} + o(1) \qquad (\epsilon \leqq t \leqq 2\epsilon). \tag{52}$$

Ausserdem ist nach (49)

$$\int_\epsilon^t (\dot{w}^* t^{4/3})^{\cdot} t^{2/3}\, dt = [\dot{w}^* t^2]_\epsilon^t - \tfrac{2}{3} \int_\epsilon^t \dot{w}^* t\, dt = o(t) \qquad (\epsilon \leqq t \leqq 2\epsilon). \tag{53}$$

Wir integrieren jetzt die Gleichung (50) zwischen den Grenzen ϵ und 2ϵ und ersetzen nachträglich wieder ϵ durch t. Nach (51), (52), (53) folgt dann

$$-\tfrac{2}{9} w_k^* t + o(t) = \frac{1}{m_k} \frac{\partial U^*}{\partial w_k^*} t + o(t)$$

$$\tfrac{2}{9} w_k^* + \frac{1}{m_k} \frac{\partial U^*}{\partial w_k^*} \to 0 \qquad (t \to 0). \tag{54}$$

Diese Relation ist offenbar invariant bei beliebiger orthogonaler Transformation des Koordinatensystems. Wir wählen ein derartiges bewegliches Koordinatensystem durch den Schwerpunkt, dass die Strecke A_3A_1 parallel zur Abszissenachse ist und A_2 eine nicht-negative Ordinate hat. Sind X_k, Y_k die neuen Koordinaten von A_k $(k = 1, 2, 3)$, so ist also $X_1 > X_3$, $Y_1 = Y_3$, $Y_2 \geqq 0$. Setzt man noch

$$p_1 = X_1 - X_3, \qquad p_2 = X_2 - X_3, \qquad p_3 = Y_2 - Y_3, \tag{55}$$

so haben im neuen System die Punkte A_1 und A_2 in bezug auf A_3 die Relativkoordinaten p_1, 0 und p_2, p_3. Es sei wieder

$$X_k^* = X_k t^{-2/3}, \qquad Y_k^* = Y_k t^{-2/3}, \qquad p_k^* = p_k t^{-2/3}.$$

Benutzt man (54) mit Y_2^*, Y_3^* anstelle von w_k^*, so folgt durch Subtraktion

$$p_3^*\left(\frac{1}{r_2^{*3}} - \frac{1}{r_3^{*3}}\right) \to 0. \tag{56}$$

Verwendet man (54) analog für X_1^*, X_3^* und für X_2^*, X_3^*, so erhält man

$$\tfrac{2}{9}p_1^* + m_2\left(\frac{p_2^* - p_1^*}{r_3^{*3}} - \frac{p_2^*}{r_1^{*3}}\right) - (m_1 + m_3)\,\frac{p_1^*}{r_2^{*3}} \to 0 \tag{57}$$

$$\tfrac{2}{9}p_2^* + m_1\left(\frac{p_1^* - p_2^*}{r_3^{*3}} - \frac{p_1^*}{r_2^{*3}}\right) - (m_2 + m_3)\,\frac{p_2^*}{r_1^{*3}} \to 0\,. \tag{58}$$

Nach (23) sind die Werte p_k^* $(k = 1, 2, 3)$ beschränkt; nach (37) sind auch die reziproken Werte der r_k^* beschränkt. Wir betrachten jetzt irgend eine Folge $t \to 0$, für welche die zugehörigen Werte der p_k^* gegen Grenzwerte $\hat{p}_k$ streben. Es seien $\hat{r}_k$ $(k = 1, 2, 3)$ die Grenzwerte der Grössen r_k^*. Da der Schwerpunkt des Dreiecks im Nullpunkt liegt, so streben auch die Punkte (X_k^*, Y_k^*) gegen gewisse Grenzpunkte $\hat{A}_k$ $(k = 1, 2, 3)$. Zufolge (56) gilt

$$\hat{p}_3(\hat{r}_2^{-3} - \hat{r}_3^{-3}) = 0,$$

also entweder $\hat{p}_3 = 0$ oder $\hat{r}_2 = \hat{r}_3$.

Liegen die drei Punkte $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$ nicht auf einer Geraden, so ist $\hat{p}_3 \neq 0$, also $\hat{r}_2 = \hat{r}_3$, und durch zyklische Vertauschung folgt auch $\hat{r}_3 = \hat{r}_1$; daher ist das Dreieck $\hat{A}_1\hat{A}_2\hat{A}_3$ dann gleichseitig. Bedeutet r die Dreiecksseite, so ist offenbar

$$\hat{p}_1 = r, \qquad \hat{p}_2 = \tfrac{1}{2}r, \qquad \hat{p}_3 = \tfrac{1}{2}r\sqrt{3}, \tag{59}$$

also nach (4) und (57)

$$\tfrac{2}{9}r^3 = m, \tag{60}$$

ferner nach (37)

$$\tfrac{2}{9}\lambda = (m_2m_3 + m_3m_1 + m_1m_2)r^{-1}. \tag{61}$$

Liegen $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$ auf einer Geraden, so kann man nach etwaiger zyklischer Vertauschung der Indizes voraussetzen, dass $\hat{A}_2$ zwischen $\hat{A}_1$ und $\hat{A}_3$ gelegen ist. Setzt man

$$\hat{p}_1 = \rho, \qquad \hat{p}_2 = \omega\rho, \tag{62}$$

so ist also

$$0 < \omega < 1 \tag{63}$$

und aus (57), (58) folgen die Gleichungen

$$\tfrac{2}{9}\rho^3 = m_1 + m_3 + m_2\{\omega^{-2} + (1 - \omega)^{-2}\} \tag{64}$$

$$\tfrac{2}{9}\omega\rho^3 = m_1\{1 - (1 - \omega)^{-2}\} + (m_2 + m_3)\omega^{-2}. \tag{65}$$

Daher genügt ω der algebraischen Gleichung fünften Grades

$$(66)\quad m_1\{(1-\omega)^{-2}-(1-\omega)\}+m_2\{\omega(1-\omega)^{-2}-(1-\omega)\omega^{-2}\}+m_3(\omega-\omega^{-2})=0.$$

Schreibt man sie in der Form

$$(67)\qquad \frac{m_1+m_2\omega}{m_1+m_2\omega^{-2}}=\frac{m_3+m_2(1-\omega)}{m_3+m_2(1-\omega)^{-2}},$$

so ist leicht zu sehen, dass sie genau eine Wurzel im Intervall (63) besitzt. Lässt man nämlich ω von 0 bis 1 wandern, so wächst die linke Seite von (67) monoton von 0 bis 1 und die rechte Seite fällt monoton von 1 bis 0. Hat man ω bestimmt, so erhält man ρ vermöge (64) und dann $\hat{p}_1$, $\hat{p}_2$ aus (62), während $\hat{p}_3 = 0$ ist. Aus (37) folgt jetzt

$$(68)\qquad \tfrac{2}{9}\lambda = \{m_2m_3\omega^{-1}+m_3m_1+m_1m_2(1-\omega)^{-1}\}\rho^{-1}.$$

Man erhält die beiden anderen geradlinigen Fälle, wenn man in (64) und (66) die Massen m_1, m_2, m_3 zyklisch vertauscht.

Aus (59), (60), (62), (64), (66) ist nun ersichtlich, dass sowohl im gleichseitigen Fall als auch in den drei geradlinigen Fällen die Grössen $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ eindeutig durch m_1, m_2, m_3 bestimmt sind. Wir hatten bisher eine solche Folge $t \to 0$ betrachtet, für welche die Werte p_k^* $(k = 1, 2, 3)$ konvergieren. Da aber die p_k^* für $t > 0$ stetige Funktionen von t sind und nur jene vier isolierten Systeme von Häufungswerten $\hat{p}_1$, $\hat{p}_2$, $\hat{p}_3$ möglich sind, so konvergieren die p_k^* auch, wenn t beliebig gegen 0 strebt. Damit ist bewiesen, dass die Ausdrücke $r_k t^{-2/3}$ $(k = 1, 2, 3)$ für $t \to 0$ gegen positive Grenzwerte streben, nämlich entweder gegen den durch (60) festgelegten Wert r des gleichseitigen Falles oder gegen die Werte $\omega\rho$, ρ, $(1 - \omega)\rho$ der drei geradlinigen Fälle, welche sich aus (64), (66) bestimmen, nach etwaiger zyklischer Vertauschung der Indizes.

Wir wollen weiterhin von den 3 geradlinigen Fällen nur noch den durch (64), (66) fixierten studieren, da die beiden anderen durch Vertauschung der Indizes auf diesen zurückgeführt werden.

7. Ein Spezialfall

Bei den bekannten von Lagrange entdeckten speziellen Lösungen des Dreikörperproblemes bewegen sich A_1, A_2, A_3 auf drei in einer Ebene gelegenen Kegelschnitten, während das Dreieck $A_1A_2A_3$ dauernd einem festen Dreieck ähnlich bleibt. Dabei ergeben sich für die Form des Dreiecks zwei Möglichkeiten: Entweder bilden A_1, A_2, A_3 die Ecken eines gleichseitigen Dreiecks oder sie liegen auf einer Geraden. Setzt man im letzteren Fall voraus, dass A_2 zwischen A_1 und A_3 gelegen ist, so erhält man als Wert des Verhältnisses der Strecken A_2A_3 und A_1A_3 gerade die durch (66) definierte Zahl ω. Hierdurch wird nahe gelegt, die unter den Lagrangeschen Lösungen enthaltenen Dreierstossbahnen aufzusuchen, um dann die allgemeinen Dreierstossbahnen mit diesen vergleichen zu können.

Man erhält die Kollisionsbahnen unter den Lagrangeschen Lösungen, indem man die kegelschnittförmige Bewegung der Massenpunkte in eine geradlinige ausarten lässt. Dementsprechend machen wir den speziellen Ansatz

$$x_k = \hat{x}_k g(t), \qquad y_k = \hat{y}_k g(t) \qquad (k = 1, 2, 3)$$

mit konstanten $\hat{x}_k$, $\hat{y}_k$ und einer zweimal differentiierbaren Funktion $g(t)$, die für $t > 0$ positiv ist und für $t = 0$ verschwindet. Versteht man unter $\hat{U}$ den Wert von U mit $\hat{x}_k$, $\hat{y}_k$ anstelle von x_k, y_k, so gehen die Differentialgleichungen (2) über in

$$(69) \qquad m_k \hat{w}_k \ddot{g} g^2 = \frac{\partial \hat{U}}{\partial \hat{w}_k} \qquad (\hat{w}_k = \hat{x}_k, \hat{y}_k;\ k = 1, 2, 3).$$

Folglich ist der Ausdruck $\ddot{g} g^2$ konstant, aber nicht 0, weil

$$\sum_{k=1}^{3} \left(\hat{x}_k \frac{\partial \hat{U}}{\partial \hat{x}_k} + \hat{y}_k \frac{\partial \hat{U}}{\partial \hat{y}_k} \right) = -\hat{U} \neq 0$$

ist. Dann kann man aber die Normierung

$$(70) \qquad \ddot{g} g^2 = -\tfrac{2}{9}$$

treffen und auf die Gleichungen (69) die Überlegungen anwenden, mit denen wir im vorigen Paragraphen die Relationen (54) untersucht haben. Wegen der orthogonalen Invarianz kann man noch $\hat{x}_1 > \hat{x}_3$, $\hat{y}_1 = \hat{y}_3$, $\hat{y}_2 \geqq 0$ annehmen und erhält für die Punkte $(\hat{x}_k, \hat{y}_k)$ $(k = 1, 2, 3)$ genau die Punkte $\hat{A}_1$, $\hat{A}_2$, $\hat{A}_3$ des gleichseitigen oder des geradlinigen Falles. Für die relativen Koordinaten $\hat{x}_1 - \hat{x}_3 = \hat{p}_1$, $\hat{x}_2 - \hat{x}_3 = \hat{p}_2$, $\hat{y}_2 - \hat{y}_3 = \hat{p}_3$ gilt dann nach (59) im gleichseitigen Falle

$$\hat{p}_1 = r, \qquad \hat{p}_2 = \tfrac{1}{2} r, \qquad \hat{p}_3 = \tfrac{1}{2} r \sqrt{3}$$

und nach (62) im geradlinigen Falle

$$\hat{p}_1 = \rho, \qquad \hat{p}_2 = \omega\rho, \qquad \hat{p}_3 = 0,$$

wobei r, ω, ρ durch (60), (66), (64) festgelegt werden.

Die Integration von (70) ergibt

$$\dot{g}^2 = \tfrac{4}{9} g^{-1} + c$$

mit konstantem c. Wählt man speziell $c = 0$, so erhält man durch nochmalige Integration

$$g = t^{2/3}$$

und damit als spezielle Dreierstosslösung

$$x_k = \hat{x}_k t^{2/3}, \qquad y_k = \hat{y}_k t^{2/3} \qquad (k = 1, 2, 3).$$

Für einen späteren Zweck berechnen wir noch die zu dieser Lösung gehörigen

Werte von $\dot{x}_1$, $\dot{x}_2$, $\dot{y}_2$. Nach (6) ist

$$\dot{x}_1 = \left(1 - \frac{m_1}{m}\right)\hat{p}_1 - \frac{m_2}{m}\hat{p}_2, \qquad \dot{x}_2 = \left(1 - \frac{m_2}{m}\right)\hat{p}_2 - \frac{m_1}{m}\hat{p}_1,$$

$$\dot{y}_2 = \left(1 - \frac{m_2}{m}\right)\hat{p}_3,$$

also im gleichseitigen Falle

$$(71) \qquad \dot{x}_1 = \frac{m_2 + 2m_3}{3m} rt^{-1/3}, \qquad \dot{x}_2 = \frac{m_3 - m_1}{3m} rt^{-1/3}, \qquad \dot{y}_2 = \frac{m_1 + m_3}{m\sqrt{3}} rt^{-1/3}$$

und im geradlinigen Falle

$$(72) \qquad \dot{x}_1 = 2\,\frac{m_2(1-\omega) + m_3}{3m}\rho t^{-1/3}, \quad \dot{x}_2 = 2\,\frac{m_3\omega - m_1(1-\omega)}{3m}\rho t^{-1/3}, \quad \dot{y}_2 = 0.$$

8. Reduktion der Differentialgleichungen

Um das Verhalten der Dreierstosslösungen bei $t = 0$ noch näher zu untersuchen, müssen wir die in (55) definierten Grössen p_1, p_2, p_3 in die Differentialgleichungen (2) einführen. Zunächst bilden wir die Relativkoordinaten von A_1 und A_2 in bezug auf A_3 im ruhenden Koordinatensystem

$$(73) \qquad \xi_1 = x_1 - x_3, \qquad \xi_2 = y_1 - y_3, \qquad \xi_3 = x_2 - x_3, \qquad \xi_4 = y_2 - y_3$$

und setzen noch

$$(74) \qquad m_1\dot{x}_1 = \eta_1, \qquad m_1\dot{y}_1 = \eta_2, \qquad m_2\dot{x}_2 = \eta_3, \qquad m_2\dot{y}_2 = \eta_4.$$

Nach (6) ist dann

$$(75) \qquad m_3\dot{x}_3 = -(\eta_1 + \eta_3), \qquad m_3\dot{y}_3 = -(\eta_2 + \eta_4),$$

also nach (7)

$$(76) \qquad T = \frac{1}{2m_1}(\eta_1^2 + \eta_2^2) + \frac{1}{2m_2}(\eta_3^2 + \eta_4^2) + \frac{1}{2m_3}\{(\eta_1 + \eta_3)^2 + (\eta_2 + \eta_4)^2\}.$$

Ferner wird

$$(77) \qquad r_1^2 = \xi_3^2 + \xi_4^2, \qquad r_2^2 = \xi_1^2 + \xi_2^2, \qquad r_3^2 = (\xi_1 - \xi_3)^2 + (\xi_2 - \xi_4)^2,$$

also U eine Funktion von ξ_1, ξ_2, ξ_3, ξ_4 allein. Die Differentialgleichungen (2) gehen dann über in das System achter Ordnung

$$(78) \qquad \left\{\begin{aligned} &\dot{\xi}_1 = \left(\frac{1}{m_1} + \frac{1}{m_3}\right)\eta_1 + \frac{1}{m_3}\eta_3, \qquad \dot{\xi}_3 = \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\eta_3 + \frac{1}{m_3}\eta_1, \\ &\dot{\xi}_2 = \left(\frac{1}{m_1} + \frac{1}{m_3}\right)\eta_2 + \frac{1}{m_3}\eta_4, \qquad \dot{\xi}_4 = \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\eta_4 + \frac{1}{m_3}\eta_2, \\ &\qquad\qquad \dot{\eta}_k = \frac{\partial U}{\partial \xi_k} \qquad (k = 1, \cdots, 4). \end{aligned}\right.$$

Führt man die Energie

$$E = T - U$$

ein, so hat das System (78) die kanonische Form

$$\dot{\xi}_k = \frac{\partial E}{\partial \eta_k}, \qquad \dot{\eta}_k = -\frac{\partial E}{\partial \xi_k} \qquad (k = 1, \cdots, 4). \tag{79}$$

Um nun die Differentialgleichungen für die Relativkoordinaten im bewegten Koordinatensystem aus §6 aufzustellen, macht man am bequemsten von der Jacobischen Transformationstheorie Gebrauch. Nach dieser gilt bekanntlich folgender Satz:

Es sei H eine Funktion von $2n$ Variabeln η_k, p_k $(k = 1, \cdots, n)$ mit stetigen partiellen Ableitungen zweiter Ordnung in der Umgebung einer Stelle, an welcher die n-reihige Determinante

$$D = \left| \frac{\partial^2 H}{\partial p_k \partial \eta_l} \right| \neq 0 \tag{80}$$

ist. Dann wird durch den Ansatz

$$\xi_k = \frac{\partial H}{\partial \eta_k}, \qquad q_k = \frac{\partial H}{\partial p_k} \qquad (k = 1, \cdots, n) \tag{81}$$

eine Variabelntransformation definiert, welche das Hamiltonsche System

$$\dot{\xi}_k = \frac{\partial E}{\partial \eta_k}, \qquad \dot{\eta}_k = -\frac{\partial E}{\partial \xi_k} \qquad (k = 1, \cdots, n)$$

in das Hamiltonsche System

$$\dot{p}_k = \frac{\partial E}{\partial q_k}, \qquad \dot{q}_k = -\frac{\partial E}{\partial p_k} \qquad (k = 1, \cdots, n)$$

überführt.

Bedeutet p_4 den Winkel zwischen der ruhenden Abszissenachse und der Richtung A_3A_1, so bestehen zwischen ξ_1, ξ_2, ξ_3, ξ_4 und den in (55) erklärten Relativkoordinaten p_1, 0, p_2, p_3 im bewegten Koordinatensystem die Gleichungen

$$\begin{aligned} \xi_1 = p_1 \cos p_4, \qquad \xi_2 = p_1 \sin p_4, \qquad \xi_3 = p_2 \cos p_4 - p_3 \sin p_4, \\ \xi_4 = p_2 \sin p_4 + p_3 \cos p_4. \end{aligned} \tag{82}$$

Wir wählen

$$\begin{aligned} H = \eta_1 p_1 \cos p_4 + \eta_2 p_1 \sin p_4 + \eta_3(p_2 \cos p_4 - p_3 \sin p_4) \\ + \eta_4(p_2 \sin p_4 + p_3 \cos p_4) \end{aligned}$$

und wenden den Transformationssatz für $n = 4$ an. Eine leichte Rechnung ergibt für die Determinante D in (80) den Wert p_1, und dieser ist von 0 ver-

schieden, solange keine Kollision vorliegt, also für $t > 0$. Die erste Gleichung in (81) ist wegen (82) erfüllt, die zweite Gleichung ergibt die Formeln

$$(83)\qquad \begin{gathered} q_1 = \eta_1 \cos p_4 + \eta_2 \sin p_4\,, \qquad q_2 = \eta_3 \cos p_4 + \eta_4 \sin p_4\,, \\ q_3 = -\eta_3 \sin p_4 + \eta_4 \cos p_4\,, \end{gathered}$$

$$(84)\qquad \begin{aligned} q_4 = -\eta_1 p_1 \sin p_4 + \eta_2 p_1 \cos p_4 - \eta_3(p_2 \sin p_4 + p_3 \cos p_4) \\ + \eta_4(p_2 \cos p_4 - p_3 \sin p_4), \end{aligned}$$

also

$$(85)\qquad q_4 = p_1(-\eta_1 \sin p_4 + \eta_2 \cos p_4) + p_2 q_3 - p_3 q_2\,.$$

Setzt man zur Abkürzung noch

$$(86)\qquad q_0 = (p_3 q_2 - p_2 q_3 + q_4) p_1^{-1},$$

so ist nach (85)

$$q_0 = -\eta_1 \sin p_4 + \eta_2 \cos p_4\,,$$

und in Verbindung mit (83) folgt

$$(87)\qquad \begin{cases} \eta_1 = q_1 \cos p_4 - q_0 \sin p_4, & \eta_2 = q_1 \sin p_4 + q_0 \cos p_4, \\ \eta_3 = q_2 \cos p_4 - q_3 \sin p_4, & \eta_4 = q_2 \sin p_4 + q_3 \cos p_4. \end{cases}$$

Daher gilt nach (76)

$$(88)\qquad T = \frac{1}{2m_1}(q_0^2 + q_1^2) + \frac{1}{2m_2}(q_2^2 + q_3^2) + \frac{1}{2m_3}\{(q_0 + q_3)^2 + (q_1 + q_2)^2\},$$

ferner nach (77) und (82)

$$(89)\qquad r_1^2 = p_2^2 + p_3^2\,, \qquad r_2 = p_1\,, \qquad r_3^2 = (p_1 - p_2)^2 + p_3^2\,.$$

Auf Grund des Transformationssatzes geht das System (79) durch die Substitutionen (82), (87) über in

$$(90)\qquad \dot{p}_k = \frac{\partial E}{\partial q_k}, \qquad \dot{q}_k = -\frac{\partial E}{\partial p_k} \qquad (k = 1, \cdots, 4),$$

wobei $E = T - U$ nach (88) und (89) als Funktion der p_k, q_k $(k = 1, \cdots, 4)$ anzusehen ist. Mit Rücksicht auf (86) ist ersichtlich, dass E die Variable p_4 nicht enthält, also

$$\frac{\partial E}{\partial p_4} = 0$$

ist. Die zweite Gleichung (90) ergibt für $k = 4$, dass q_4 konstant ist. Dies ist gerade die Aussage des Flächenintegrals, denn aus (73), (74), (75), (82), (84) folgt

$$q_4 = -\eta_1 \xi_2 + \eta_2 \xi_1 - \eta_3 \xi_4 + \eta_4 \xi_3 = \sum_{k=1}^{3} m_k(x_k \dot{y}_k - y_k \dot{x}_k);$$

also ist q_4 die Konstante γ_3 aus (5). Nach §5 ist daher für jede Dreierstossbahn

$$q_4 = 0, \tag{91}$$

und wir erhalten für p_k, q_k $(k = 1, 2, 3)$ das System sechster Ordnung

$$\dot{p}_k = \left(\frac{\partial E}{\partial q_k}\right)_{q_4=0}, \qquad \dot{q}_k = -\left(\frac{\partial E}{\partial p_k}\right)_{q_4=0} \qquad (k = 1, 2, 3).$$

Ist dieses integriert, so ergibt sich p_4 durch Quadratur aus

$$\dot{p}_4 = \left(\frac{\partial E}{\partial q_4}\right)_{q_4=0}.$$

9. Das asymptotische Verhalten der p_k und q_k

Wir haben in §6 bewiesen, dass die Ausdrücke $p_k^* = p_k t^{-2/3}$ $(k = 1, 2, 3)$ für $t \to 0$ Grenzwerte $\hat{p}_k$ haben. Nach (59) ist im gleichseitigen Falle

$$p_1 \sim r t^{2/3}, \qquad p_2 \sim \tfrac{1}{2} r t^{2/3}, \qquad p_3 \sim \frac{\sqrt{3}}{2} r t^{2/3}, \tag{92}$$

wobei r durch (60) gegeben ist, und nach (62) im geradlinigen Falle

$$p_1 \sim \rho t^{2/3}, \qquad p_2 \sim \omega\rho t^{2/3}, \qquad p_3 = o(t^{2/3}), \tag{93}$$

wobei die Zahlen ω, ρ in (66), (64) festgelegt sind.

Wir wenden uns jetzt zur Untersuchung des asymptotischen Verhaltens von q_1, q_2, q_3. Schreibt man zur Abkürzung

$$\cos p_4 = \mu, \qquad \sin p_4 = \nu, \tag{94}$$

so ist nach (74), (75), (87)

$$\left\{\begin{aligned} &m_1\dot{x}_1 = q_1\mu - q_0\nu, \qquad m_2\dot{x}_2 = q_2\mu - q_3\nu,\\ &\qquad m_3\dot{x}_3 = -(q_1 + q_2)\mu + (q_0 + q_3)\nu,\\ &m_1\dot{y}_1 = q_1\nu + q_0\mu, \qquad m_2\dot{y}_2 = q_2\nu + q_3\mu,\\ &\qquad m_3\dot{y}_3 = -(q_1 + q_2)\nu - (q_0 + q_3)\mu. \end{aligned}\right. \tag{95}$$

Nach (6), (73), (82) ist ferner

$$\left\{\begin{aligned} &mx_1 = \{(m_2 + m_3)p_1 - m_2p_2\}\mu + m_2p_3\nu,\\ &\qquad\qquad my_1 = \{(m_2 + m_3)p_1 - m_2p_2\}\nu - m_2p_3\mu,\\ &mx_2 = \{(m_1 + m_3)p_2 - m_1p_1\}\mu - (m_1 + m_3)p_3\nu,\\ &\qquad\qquad my_2 = \{(m_1 + m_3)p_2 - m_1p_1\}\nu + (m_1 + m_3)p_3\mu,\\ &mx_3 = -(m_1p_1 + m_2p_2)\mu + m_2p_3\nu, \quad my_3 = -(m_1p_1 + m_2p_2)\nu - m_2p_3\mu. \end{aligned}\right. \tag{96}$$

Hieraus folgt mit Rücksicht auf (10)

$$\tfrac{1}{2}\dot{J} = p_1q_1 + p_2q_2 + p_3q_3\,,$$

also nach (25)

$$p_1q_1 + p_2q_2 + p_3q_3 \sim \tfrac{2}{3}\lambda t^{1/3}, \tag{97}$$

wobei λ im gleichseitigen Falle durch (61), im geradlinigen Falle durch (68) festgelegt wird. Ausserdem ist nach (38)

$$x_1\dot{y}_1 - y_1\dot{x}_1 = o(t^{1/3}), \qquad x_2\dot{y}_2 - y_2\dot{x}_2 = o(t^{1/3}),$$

also

$$\{(m_2 + m_3)p_1 - m_2p_2\}q_0 + m_2p_3q_1 = o(t^{1/3}), \tag{98}$$

$$\{(m_1 + m_3)p_2 - m_1p_1\}q_3 - (m_1 + m_3)p_3q_2 = o(t^{1/3}). \tag{99}$$

Zufolge (43), (83), (86) ist

$$q_k = O(t^{-1/3}) \qquad (k = 0, \cdots, 3). \tag{100}$$

Im gleichseitigen Falle folgen aus (91), (92), (97), (98), (99), (100) die Beziehungen

$$2q_1 + q_2 + q_3\sqrt{3} \sim \tfrac{4}{3}\lambda r^{-1}t^{-1/3}, \tag{101}$$

$$(m_2 + 2m_3)(q_2\sqrt{3} - q_3) + 2m_2q_1\sqrt{3} = o(t^{-1/3}), \tag{102}$$

$$(m_3 - m_1)q_3 - (m_1 + m_3)q_2\sqrt{3} = o(t^{-1/3}). \tag{102}$$

Setzt man

$$q_3 = \frac{1}{\sqrt{3}}\, m_2(m_1 + m_3)q,$$

so wird nach (103)

$$q_2 = \tfrac{1}{3}m_2(m_3 - m_1)q + o(t^{-1/3})$$

und nach (102)

$$q_1 = \tfrac{1}{3}m_1(m_2 + 2m_3)q + o(t^{-1/3}),$$

also nach (101)

$$\tfrac{4}{3}(m_2m_3 + m_3m_1 + m_1m_2)q \sim \tfrac{4}{3}\lambda r^{-1}t^{-1/3},$$

woraus nach (60), (61)

$$q \sim \frac{r}{m}\, t^{-1/3}$$

und folglich

$$q_1 \sim \frac{m_1}{3m}(m_2 + 2m_3)rt^{-1/3}, \qquad q_2 \sim \frac{m_2}{3m}(m_3 - m_1)rt^{-1/3},$$

(104)

$$q_3 \sim \frac{m_2}{m\sqrt{3}}(m_1 + m_3)rt^{-1/3}$$

sich ergibt.

Im geradlinigen Falle ist nach (86), (91), (93), (98), (100)

$$-\{m_2(1-\omega) + m_3\}\omega q_3 = o(t^{-1/3}),$$

also

(105) $$q_3 = o(t^{-1/3}), \qquad q_0 = o(t^{-1/3}).$$

Nach (38) ist ferner

$$m_1 m_2(x_1\dot{x}_2 - x_2\dot{x}_1 + y_1\dot{y}_2 - y_2\dot{y}_1) = o(t^{1/3}),$$

also nach (95), (96), (105)

(106) $$m_1\{m_2(1-\omega) + m_3\}q_2 + m_2\{m_1(1-\omega) - m_3\omega\}q_1 = o(t^{1/3}).$$

Setzt man diesmal

$$q_1 = m_1\{m_2(1-\omega) + m_3\}q,$$

so wird

$$q_2 = m_2\{m_3\omega - m_1(1-\omega)\}q + o(t^{1/3})$$

und nach (97)

(107) $$\{m_2 m_3\omega^2 + m_3 m_1 + m_1 m_2(1-\omega)^2\}q \sim \tfrac{2}{3}\lambda\rho^{-1}t^{-1/3}.$$

Multipliziert man (64) mit $m_1\{m_2(1-\omega) + m_3\}$ und (65) mit $m_2\{m_3\omega - m_1(1-\omega)\}$, so folgt durch Addition

$$\tfrac{2}{9}\{m_2 m_3\omega^2 + m_3 m_1 + m_1 m_2(1-\omega)^2\}\rho^3 = \{m_2 m_3\omega^{-1} + m_3 m_1 + m_1 m_2(1-\omega)^{-1}\}m.$$

Benutzt man noch (68), so geht (107) über in

$$q \sim \frac{2}{3}\frac{\rho}{m}t^{-1/3}.$$

Demnach ist jetzt

$$q_1 \sim \frac{2m_1}{3m}\{m_2(1-\omega) + m_3\}\rho t^{-1/3},$$

(108)

$$q_2 \sim \frac{2m_2}{3m}\{m_3\omega - m_1(1-\omega)\}\rho t^{-1/3}, \qquad q_3 = o(t^{-1/3}).$$

Durch (104) und (108) ist das asymptotische Verhalten von q_1, q_2, q_3 im gleichseitigen und im geradlinigen Falle festgestellt. Dass auch p_4 einen Grenzwert für $t \to 0$ hat, wird sich erst im späteren Verlauf der Untersuchung ergeben.

Man kann (104) auch mit etwas geringerer Rechnung aus den Formeln (101), (102), (103) erhalten, indem man von §7 Gebrauch macht. Da nämlich die auf den linken Seiten dieser Formeln stehenden linearen Formen von q_1, q_2, q_3 linear unabhängig sind, so sind die Grenzwerte von $q_k t^{1/3}$ $(k = 1, 2, 3)$ für $t \to 0$ eindeutig bestimmt. Also kann man sie durch Betrachtung des speziellen Falles von §7 ermitteln. Dort ist aber $p_4 = 0$ und folglich nach (95)

$$q_1 = m_1 \dot{x}_1, \qquad q_2 = m_2 \dot{x}_2, \qquad q_3 = m_2 \dot{y}_2.$$

Aus (71) folgt dann (104). Analog kann man im geradlinigen Falle aus (72), (97), (105), (106) ohne weitere Rechnung auf (108) schliessen.

10. Die charakteristische Gleichung

Wir machen in den Differentialgleichungen (90) die Substitutionen

$$\begin{cases} p_k = p_k^* t^{2/3}, & q_k = q_k^* t^{-1/3} \quad (k = 1, 2, 3), \\ p_4 = p_4^*, & q_4 = q_4^* t^{1/3}, \end{cases} \tag{109}$$

$$t = e^{-s}. \tag{110}$$

Nach (86), (88), (89) ist E eine Funktion von p_k, q_k $(k = 1, \cdots, 4)$, in welcher p_4 nicht auftritt. Ersetzt man darin die Variabeln p_k, q_k durch p_k^*, q_k^*, so möge E^* entstehen, und zwar ist

$$E^* = Et^{2/3}.$$

Das System (90) geht dadurch über in

$$\begin{cases} \dfrac{dp_k^*}{ds} = \tfrac{2}{3}p_k^* - \dfrac{\partial E^*}{\partial q_k^*}, & \dfrac{dq_k^*}{ds} = -\tfrac{1}{3}q_k^* + \dfrac{\partial E^*}{\partial p_k^*} \quad (k = 1, 2, 3), \\ \dfrac{dp_4^*}{ds} = -\dfrac{\partial E^*}{\partial q_4^*}, & \dfrac{dq_4^*}{ds} = \tfrac{1}{3}q_4^* + \dfrac{\partial E}{\partial p_4^*}. \end{cases} \tag{111}$$

Für $t \to 0$ ist $s \to \infty$. Wie im vorigen Paragraphen gezeigt wurde, haben bei diesem Grenzübergang die Ausdrücke p_k^* und q_k^* $(k = 1, 2, 3)$ bestimmte Grenzwerte $\hat{p}_k$ und $\hat{q}_k$. Im gleichseitigen Falle ist nach (92) und (104)

$$\begin{cases} \hat{p}_1 = r, \qquad \hat{p}_2 = \tfrac{1}{2}r, \qquad \hat{p}_3 = \tfrac{1}{2}r\sqrt{3}, \\ \hat{q}_1 = \dfrac{m_1}{3m}(m_2 + 2m_3)r, \qquad \hat{q}_2 = \dfrac{m_2}{3m}(m_3 - m_1), \\ \hat{q}_3 = \dfrac{m_2}{m\sqrt{3}}(m_1 + m_3)r, \end{cases} \tag{112}$$

im geradlinigen Falle nach (93) und (108)

$$(113)\begin{cases} \hat{p}_1 = \rho, \quad \hat{p}_2 = \omega\rho, \quad \hat{p}_3 = 0, \\ \hat{q}_1 = \dfrac{2m_1}{3m}\{m_2(1-\omega) + m_3\}\rho, \quad \hat{q}_2 = \dfrac{2m_2}{3m}\{m_3\omega - m_1(1-\omega)\}\rho, \\ \hat{q}_3 = 0. \end{cases}$$

Nach §7 kennen wir eine spezielle Dreierstosslösung, nämlich

$$p_k = \hat{p}_k t^{2/3}, \quad q_k = \hat{q}_k t^{-1/3} \ (k = 1, 2, 3), \quad p_4 = 0, \quad q_4 = 0,$$

wobei also $\hat{p}_k$, $\hat{q}_k$ $(k = 1, 2, 3)$ im gleichseitigen Falle durch (112), im geradlinigen Falle durch (113) gegeben sind. Wir erhalten also eine spezielle Lösung des Systems (111), eine Gleichgewichtslösung, wenn wir p_k^*, q_k^* konstant gleich $\hat{p}_k, \hat{q}_k$ $(k = 1, 2, 3)$ und $p_4 = 0$, $q_4 = 0$ setzen. Um sämtliche Dreierstossbahnen zu bekommen, haben wir sämtliche Lösungen von (111) zu untersuchen, für welche p_k^*, q_k^* $(k = 1, 2, 3)$ die Grenzwerte $\hat{p}_k$, $\hat{q}_k$ haben, wenn s über alle Grenzen wächst.

Wir setzen noch

$$(114)\quad p_k^* = \hat{p}_k + \delta_k, \quad q_k^* = \hat{q}_k + \delta_{k+3} \ (k = 1, 2, 3), \quad p_4^* = \delta_8, \quad q_4^* = \delta_7.$$

Für genügend kleine Werte der absoluten Beträge von δ_1, δ_2, δ_3 lässt sich E^* in eine Reihe nach Potenzen von $\delta_1, \cdots, \delta_7$ entwickeln, deren Koeffizienten noch von m_1, m_2, m_3 abhängen. Die Differentialgleichungen (111) gehen dadurch über in ein System der Gestalt

$$(115)\quad \frac{d\delta_k}{ds} = \sum_{l=1}^{8} a_{kl}\delta_l + \varphi_k \qquad (k = 1, \cdots, 8),$$

wo a_{kl} $(k, l = 1, \cdots, 8)$ Konstante bedeuten und $\varphi_1, \cdots, \varphi_8$ Potenzreihen in $\delta_1, \cdots, \delta_8$ ohne konstante und lineare Glieder. Da E^* nicht die Variable p_4^* enthält, so gilt nach (111)

$$(116)\quad a_{k8} = 0 \ (k = 1, \cdots, 8), \quad a_{77} = \tfrac{1}{3}, \quad a_{7l} = 0 \ (l \neq 7), \quad \varphi_7 = 0.$$

Es bedeute $\mathfrak{M}$ die achtreihige Matrix aus den Elementen a_{kl}, ferner $\mathfrak{A}$ die Untermatrix von $\mathfrak{M}$, die durch Streichung der beiden letzten Zeilen und Spalten aus $\mathfrak{M}$ hervorgeht. Wird eine Einheitsmatrix mit $\mathfrak{E}$ bezeichnet, so sind zufolge (116) die charakteristischen Polynome

$$G(z) = |z\mathfrak{E} - \mathfrak{M}|, \qquad F(z) = |z\mathfrak{E} - \mathfrak{A}|$$

der Matrizen $\mathfrak{M}$ und $\mathfrak{A}$ durch die Gleichung

$$(117)\quad G(z) = z(z - \tfrac{1}{3})F(z)$$

verknüpft.

Für jede Dreierstosslösung ist $q_4 = 0$, also $\delta_7 = 0$. Es sei ψ_k der Wert von φ_k für $\delta_7 = 0$. Wir haben dann das System

$$\frac{d\delta_k}{ds} = \sum_{l=1}^{6} a_{kl}\delta_l + \psi_k \qquad (k = 1, \cdots, 6), \tag{118}$$

$$\frac{d\delta_8}{ds} = \sum_{l=1}^{6} a_{8l}\delta_l + \psi_8 \tag{119}$$

zu lösen, unter der Bedingung $\delta_k \to 0$ $(k = 1, \cdots, 6)$ für $s \to \infty$. Auf den rechten Seiten dieser 7 Differentialgleichungen tritt δ_8 nicht auf. Hat man also das System (118) vollständig gelöst, unter jener Bedingung, so ergibt sich $\delta_8 = p_4$ aus (119) durch eine Quadratur. Zur näheren Diskussion von (118) ist die Kenntnis der charakteristischen Wurzeln der Matrix $\mathfrak{A}$ notwendig. Die direkte Berechnung der Determinante $| z\mathfrak{E} - \mathfrak{A} |$ ist recht mühsam, da die Koeffizienten a_{kl} sich nicht bequem bestimmen lassen. Einfacher erhält man $F(z)$ durch folgende Überlegung:

Bezeichnet man die rechten Seiten der Differentialgleichungen (115) mit Φ_k $(k = 1, \cdots, 8)$, so hat man das System

$$\frac{d\delta_k}{ds} = \Phi_k \qquad (k = 1, \cdots, 8) \tag{120}$$

und $\mathfrak{M}$ ist die Funktionalmatrix $\left(\frac{\partial \Phi_k}{\partial \delta_l}\right)$ an der Stelle $\delta_1 = 0, \cdots, \delta_8 = 0$. Man betrachte jetzt $\delta_1, \cdots, \delta_8$ als zweimal stetig differentiierbare Funktionen von 8 neuen Variabeln $\theta_1, \cdots, \theta_8$, und zwar möge für das Wertsystem $\theta_k = \hat{\theta}_k$ $(k = 1, \cdots, 8)$ speziell $\delta_k = 0$ $(k = 1, \cdots, 8)$ sein und die Funktionaldeterminante der δ_k bezüglich der Variabeln θ_k an der Stelle $\theta_k = \hat{\theta}_k$ nicht verschwinden. Durch diese Transformation gehen die Differentialgleichungen (120) über in

$$\frac{d\theta_k}{ds} = \sum_{g=1}^{8} \frac{\partial \theta_k}{\partial \delta_g} \Phi_g \qquad (k = 1, \cdots, 8),$$

wobei die rechten Seiten als Funktionen der θ_k anzusehen sind. Bezeichnet man diese rechten Seiten zur Abkürzung mit Ψ_k $(k = 1, \cdots, 8)$, so ist

$$\frac{\partial \Psi_k}{\partial \theta_l} = \sum_{g=1}^{8} \Phi_g \frac{\partial}{\partial \theta_l}\left(\frac{\partial \theta_k}{\partial \delta_g}\right) + \sum_{g,h=1}^{8} \frac{\partial \theta_k}{\partial \delta_g} \frac{\partial \Phi_g}{\partial \delta_h} \frac{\partial \delta_h}{\partial \theta_l}. \tag{121}$$

Es bedeute $\mathfrak{P}$ die Funktionalmatrix $\left(\frac{\partial \Psi_k}{\partial \theta_l}\right)$ an der Stelle $\theta_1 = \hat{\theta}_1, \cdots, \theta_8 = \hat{\theta}_8$ und $\mathfrak{G}$ die Funktionalmatrix $\left(\frac{\partial \delta_k}{\partial \theta_l}\right)$ an derselben Stelle. Da der erste Summand auf der rechten Seite von (121) an dieser Stelle verschwindet, so ergibt (121) die Beziehung

$$\mathfrak{P} = \mathfrak{G}^{-1}\mathfrak{M}\mathfrak{G}$$

und folglich ist

$$| z\mathfrak{E} - \mathfrak{P} | = | z\mathfrak{E} - \mathfrak{G}^{-1}\mathfrak{M}\mathfrak{G} | = | \mathfrak{G}^{-1}(z\mathfrak{E} - \mathfrak{M})\mathfrak{G} | = | z\mathfrak{E} - \mathfrak{M} | = G(z). \tag{122}$$

Um diese Formel zur Berechnung von $G(z)$ anzuwenden, setzen wir noch

$$(123)\qquad \xi_k = \xi_k^* t^{-2/3}, \qquad \eta_k = \eta_k^* t^{-1/3} \qquad (k = 1, \cdots, 4),$$

$$q_0 = q_0^* t^{-1/3}, \qquad \hat{q}_0 = (\hat{p}_3\hat{q}_2 - \hat{p}_2\hat{q}_3)\hat{p}_1^{-1}.$$

Nach (82), (86), (87), (109) ist dann

$$\xi_1^* = \mu p_1^*, \qquad \xi_2^* = \nu p_1^*, \qquad \xi_3^* = \mu p_2^* - \nu p_3^*, \qquad \xi_4^* = \nu p_2^* + \mu p_3^*,$$

$$\eta_1^* = \mu q_1^* - \nu q_0^*, \quad \eta_2^* = \nu q_1^* + \mu q_0^*, \quad \eta_3^* = \mu q_2^* - \nu q_3^*, \quad \eta_4^* = \nu q_2^* + \mu q_3^*,$$

wobei μ, ν in (94) erklärt sind. Zufolge (114) sind die ξ_k^*, η_k^* Funktionen von $\delta_1, \cdots, \delta_8$, die an der Stelle $\delta_1 = 0, \cdots, \delta_8 = 0$ die Werte

$$_1 = \hat{p}_1, \quad \hat{\xi}_2 = 0, \quad \hat{\xi}_3 = \hat{p}_2, \quad \hat{\xi}_4 = \hat{p}_3, \quad \hat{\eta}_1 = \hat{q}_1, \quad \hat{\eta}_2 = \hat{q}_0, \quad \hat{\eta}_3 = \hat{q}_2, \quad \hat{\eta}_4 = \hat{q}_3$$

haben. Endlich sei

$$(124)\begin{cases} \theta_k = \xi_k^* \qquad (k = 1, \cdots, 4), \\ \theta_5 = -\left(\dfrac{1}{m_1} + \dfrac{1}{m_3}\right)\eta_1^* - \dfrac{1}{m_3}\eta_3^*, \qquad \theta_6 = -\left(\dfrac{1}{m_1} + \dfrac{1}{m_3}\right)\eta_2^* - \dfrac{1}{m_3}\eta_4^*, \\ \theta_7 = -\left(\dfrac{1}{m_2} + \dfrac{1}{m_3}\right)\eta_3^* - \dfrac{1}{m_3}\eta_1^*, \qquad \theta_8 = -\left(\dfrac{1}{m_2} + \dfrac{1}{m_3}\right)\eta_4^* - \dfrac{1}{m_3}\eta_2^*. \end{cases}$$

Durch eine einfache Rechnung erhält man für die Funktionaldeterminante der θ_k als Funktionen der δ_k den Wert $m^2(m_1m_2m_3)^{-2} \neq 0$. Vermöge der Substitutionen (123), (124) gehen nun die Differentialgleichungen (78) über in

$$(125)\qquad \frac{d\theta_k}{ds} = \tfrac{2}{3}\theta_k + \theta_{k+4}, \qquad \frac{d\theta_{k+4}}{ds} = -\tfrac{1}{3}\theta_{k+4} - G_k \qquad (k = 1, \cdots, 4)$$

mit

$$G_1 = m_2\theta_3R_1 + (m_1 + m_3)\theta_1R_2 + m_2(\theta_1 - \theta_3)R_3,$$

$$G_2 = m_2\theta_4R_1 + (m_1 + m_3)\theta_2R_2 + m_2(\theta_2 - \theta_4)R_3,$$

$$G_3 = m_1\theta_1R_1 + (m_2 + m_3)\theta_3R_2 + m_1(\theta_3 - \theta_1)R_3,$$

$$G_4 = m_1\theta_2R_1 + (m_2 + m_3)\theta_4R_2 + m_1(\theta_4 - \theta_2)R_3,$$

$$R_1 = (\theta_3^2 + \theta_4^2)^{-3/2}, \quad R_2 = (\theta_1^2 + \theta_2^2)^{-3/2}, \quad R_3 = \{(\theta_1 - \theta_3)^2 + (\theta_2 - \theta_4)^2\}^{-3/2}.$$

Für die Werte der Ableitungen

$$\frac{\partial G_k}{\partial \theta_l} = c_{kl} \qquad (k, l = 1, \cdots, 4)$$

an der Stelle

$$(126)\qquad \theta_1 = \hat{p}_1, \qquad \theta_2 = 0, \qquad \theta_3 = \hat{p}_2, \qquad \theta_4 = \hat{p}_3$$

findet man nach (112) im gleichseitigen Falle

$$r^3 c_{kl} = \begin{cases} \tfrac{1}{4}m_2 - 2(m_1+m_3),\ \dfrac{3\sqrt{3}}{4}\,m_2,\ 0,\ -\dfrac{3\sqrt{3}}{2}\,m_2, \\[2mm] \dfrac{3\sqrt{3}}{4}\,m_2,\ m_1+m_3-\tfrac{5}{4}m_2,\ -\dfrac{3\sqrt{3}}{2}\,m_2,\ 0, \\[2mm] -\tfrac{9}{4}m_1,\ -\dfrac{3\sqrt{3}}{4}\,m_1,\ \tfrac{1}{4}(m_1+m_2+m_3),\ \dfrac{3\sqrt{3}}{4}\,(m_1-m_2-m_3), \\[2mm] -\dfrac{3\sqrt{3}}{4}\,m_1,\ \tfrac{9}{4}m_1,\ \dfrac{3\sqrt{3}}{4}\,(m_1-m_2-m_3),\ -\tfrac{5}{4}(m_1+m_2+m_3), \end{cases}$$

und nach (113) im geradlinigen Falle

$$\rho^3 c_{kl} = \begin{cases} -2(m_1+m_3) - 2m_2\omega^{-3},\ 0,\ 2m_2\{\omega^{-3} - (1-\omega)^{-3}\},\ 0, \\ 0,\ m_1+m_3+m_2\omega^{-3},\ 0,\ m_2\{(1-\omega)^{-3} - \omega^{-3}\}, \\ -2m_1(1-\omega^{-3}),\ 0,\ -2m_1\omega^{-3} - 2(m_2+m_3)(1-\omega)^{-3},\ 0, \\ 0,\ m_1(1-\omega^{-3}),\ 0,\ m_1\omega^{-3} + (m_2+m_3)(1-\omega)^{-3}, \end{cases}$$

wo die Grössen r, ω, ρ durch (60), (66), (64) fixiert sind.

Bezeichnet man zur Abkürzung die vierreihige Matrix (c_{kl}) mit $\mathfrak{C}$, so hat die Funktionalmatrix der rechten Seiten von (125) als Funktionen von $\theta_1, \cdots, \theta_8$ an der Stelle (126) die Gestalt

$$\mathfrak{P} = \begin{pmatrix} \tfrac{2}{3}\mathfrak{E} & \mathfrak{E} \\ -\mathfrak{C} & -\tfrac{1}{3}\mathfrak{E} \end{pmatrix},$$

wo die rechts auftretenden Matrizen vierreihig sind, und nach (122) wird

$$G(z) = \begin{vmatrix} (z-\tfrac{2}{3})\mathfrak{E} & -\mathfrak{E} \\ \mathfrak{C} & (z+\tfrac{1}{3})\mathfrak{E} \end{vmatrix} = |\,(z+\tfrac{1}{3})(z-\tfrac{2}{3})\mathfrak{E} + \mathfrak{C}\,|.$$

Diese vierreihige Determinante bestimmt man durch direkte Rechnung unter Benutzung der angegebenen Werte der c_{kl}. Es ergibt sich ein einfaches Resultat. Setzt man

$$(z+\tfrac{1}{3})(z-\tfrac{2}{3}) = x,$$

so wird im gleichseitigen Falle

(127) $$G(z) = (x+\tfrac{2}{9})(x-\tfrac{4}{9})(x^2 - \tfrac{2}{9}x - \tfrac{8}{81} + \tfrac{1}{3}a)$$

mit

(128) $$a = \frac{m_2m_3 + m_3m_1 + m_1m_2}{(m_1+m_2+m_3)^2}$$

und im geradlinigen Falle

$$G(z) = (x + \tfrac{2}{9})(x - \tfrac{4}{9})(x + \tfrac{2}{9} + \tfrac{8}{9}b)(x - \tfrac{4}{9} - \tfrac{4}{9}b) \tag{129}$$

mit

$$b = \frac{m_1\{1 + (1 - \omega)^{-1} + (1 - \omega)^{-2}\} + m_3(1 + \omega^{-1} + \omega^{-2})}{m_1 + m_2\{\omega^{-2} + (1 - \omega)^{-2}\} + m_3}. \tag{130}$$

Da

$$z(z - \tfrac{1}{3}) = x + \tfrac{2}{9}$$

ist, so erhalten wir nach (117) das charakteristische Polynom $F(z)$ der Matrix $\mathfrak{A}$, indem wir auf den rechten Seiten von (127) und (129) den Faktor $x + \frac{2}{9}$ fortlassen.

Die charakteristischen Wurzeln von $\mathfrak{A}$ sind also zufolge (127) im gleichseitigen Falle die 6 Zahlen

$$\begin{cases} -a_0 = -\tfrac{2}{3}, & -a_1 = \tfrac{1}{6}\{1 - [13 + 12(1 - 3a)^{\frac{1}{2}}]^{\frac{1}{2}}\}, \\ -a_2 = \tfrac{1}{6}\{1 - [13 - 12(1 - 3a)^{\frac{1}{2}}]^{\frac{1}{2}}\}, & a_3 = \tfrac{1}{6}\{1 + [13 - 12(1 - 3a)^{\frac{1}{2}}]^{\frac{1}{2}}\}, \\ a_4 = \tfrac{1}{6}\{1 + [13 + 12(1 - 3a)^{\frac{1}{2}}]^{\frac{1}{2}}\}, & a_5 = 1. \end{cases} \tag{131}$$

Wegen der Beziehung

$$2m^2(1 - 3a) = (m_2 - m_3)^2 + (m_3 - m_1)^2 + (m_1 - m_2)^2$$

ist $a \leqq \frac{1}{3}$, und zwar $a = \frac{1}{3}$ nur für $m_1 = m_2 = m_3$; andererseits ist $a > 0$. Folglich sind die 6 Wurzeln sämtlich reell, und zwar $-a_0$, $-a_1$, $-a_2$ negativ, a_3, a_4, a_5 positiv. Ferner sind sie sämtlich verschieden, ausser im Falle $m_1 = m_2 = m_3$, wo $-a_1 = -a_2$ und $a_3 = a_4$ wird.

Im geradlinigen Falle ergeben sich aus (129) die Wurzeln

$$\begin{cases} -b_0 = -\tfrac{2}{3}, & -b_1 = \tfrac{1}{6}[1 - (25 + 16b)^{\frac{1}{2}}], & b_2 = \tfrac{1}{6}[1 - (1 - 8b)^{\frac{1}{2}}], \\ b_3 = \tfrac{1}{6}[1 + (1 - 8b)^{\frac{1}{2}}], & b_4 = \tfrac{1}{6}[1 + (25 + 16b)^{\frac{1}{2}}], & b_5 = 1. \end{cases} \tag{132}$$

Von diesen sind $-b_0$ und $-b_1$ negativ, b_4 und b_5 positiv, b_2 und b_3 entweder positiv oder konjugiert komplex mit dem positiven Realteil $\frac{1}{6}$. Die Wurzeln sind alle verschieden, ausser im Falle $b = \frac{1}{8}$, wo $b_2 = b_3 = \frac{1}{6}$ wird. Die Gleichung $b = \frac{1}{8}$ liefert eine algebraische Bedingung für m_1, m_2, m_3, die nicht identisch erfüllt ist. Wählt man z.B. $m_1 = m_3$, so ergibt (66) den Wert $\omega = \frac{1}{2}$, und aus der Annahme $b = \frac{1}{8}$ folgt nach (130) die Bedingung $\dfrac{m_2}{m_1} = \dfrac{55}{4}$. Also sind im allgemeinen die Wurzeln sämtlich verschieden.

Dass die Werte $-\frac{2}{3}$ und 1 unter den Wurzeln auftreten, hätte man auch ohne Rechnung aus dem Energieintegral entnehmen können. Dagegen lässt sich die Bestimmung der übrigen Wurzeln wohl kaum einfacher durchführen, als es hier geschehen ist.

11. Asymptotische Bahnen

Wir betrachten ein System von Differentialgleichungen der Form

$$\frac{d\delta_k}{ds} = \sum_{l=1}^{n} a_{kl}\delta_l + \psi_k \qquad (k = 1, \cdots, n). \tag{133}$$

Hierin seien die a_{kl} reelle Konstante und die ψ_k Potenzreihen der Variabeln $\delta_1, \cdots, \delta_n$ mit reellen Koeffizienten, welche in einer gewissen Umgebung des Nullpunktes konvergieren und weder konstante noch lineare Glieder enthalten. Es seien $\lambda_1, \cdots, \lambda_n$ die charakteristischen Wurzeln der Matrix $(a_{kl}) = \mathfrak{A}$ und ρ_k der reelle Teil von λ_k $(k = 1, \cdots, n)$. Von diesen reellen Teilen seien p negativ und $n - p$ positiv, also keiner gleich 0. Wir denken uns die Wurzeln so angeordnet, dass

$$0 > \rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_p, \qquad \rho_{p+1} \geqq \rho_{p+2} \geqq \cdots \geqq \rho_n > 0 \tag{134}$$

ist.

Nach einem bekannten Satze der Elementarteilertheorie gibt es eine Matrix $\mathfrak{H} = (h_{kl})$, sodass die Matrix

$$\mathfrak{H}^{-1}\mathfrak{A}\mathfrak{H} = \mathfrak{Q} \tag{135}$$

die Normalform besitzt. Zunächst werde angenommen, dass die Wurzeln λ_k $(k = 1, \cdots, n)$ sämtlich verschieden sind. Dann ist $\mathfrak{Q}$ die Diagonalmatrix aus den Diagonalelementen $\lambda_1, \cdots, \lambda_n$. Man kann noch voraussetzen, dass für je zwei konjugiert komplexe Wurzeln λ_j und λ_k auch die j^{te} und die k^{te} Spalte von $\mathfrak{H}$ zueinander konjugiert komplex sind. Macht man die Substitution

$$\delta_k = \sum_{l=1}^{n} h_{kl}\zeta_l \qquad (k = 1, \cdots, n), \tag{136}$$

so sind $\delta_1, \cdots, \delta_n$ dann und nur dann reell, wenn für jedes Paar konjugiert komplexer Wurzeln $\lambda_j, \lambda_k = \bar{\lambda}_j$ auch stets $\zeta_k = \bar{\zeta}_j$ gilt und für reelles λ_k auch ζ_k reell ist. Wir wollen weiterhin nur solche Werte der Variabeln ζ_k betrachten, die diesen Bedingungen genügen.

Durch die lineare Substitution (136) geht das System (133) wegen (135) über in

$$\frac{d\zeta_k}{ds} = \lambda_k\zeta_k + \chi_k \qquad (k = 1, \cdots, n); \tag{137}$$

dabei sind die $\chi_k = \chi_k(\zeta_1, \cdots, \zeta_n)$ Potenzreihen in $\zeta_1, \cdots, \zeta_n$ ohne konstante und lineare Glieder, welche den Gleichungen

$$\psi_k = \sum_{l=1}^{n} h_{kl}\chi_l \qquad (k = 1, \cdots, n)$$

genügen. Für $\lambda_k = \bar{\lambda}_j$ ist also auch $\chi_k = \bar{\chi}_j$.

Wir wollen das System (137) weiter vereinfachen durch Substitutionen der Gestalt

$$u_k = \zeta_k - P_k(\zeta_1, \cdots, \zeta_p) \qquad (k = 1, \cdots, n), \tag{138}$$

wo die P_k Potenzreihen der ersten p Variabeln $\zeta_1, \cdots, \zeta_p$ ohne konstante und lineare Glieder bedeuten. Wir setzen diese Potenzreihen zunächst mit unbestimmten Koeffizienten an und wollen diese dann rekursiv eindeutig festlegen durch gewisse Bedingungen. Die Untersuchung der Konvergenz werden wir nachträglich durchführen. Führt man die Variabeln u_k in die Differentialgleichungen (137) ein, so erhält man das System

$$\frac{du_k}{ds} = \lambda_k u_k + Q_k(u_1, \cdots, u_n) \qquad (k = 1, \cdots, n) \tag{139}$$

mit

$$Q_k = \chi_k + \lambda_k P_k - \sum_{l=1}^{p} \frac{\partial P_k}{\partial \zeta_l} (\chi_l + \lambda_l \zeta_l),$$

wobei in $\chi_k, P_k, \dfrac{\partial P_k}{\partial \zeta_l}, \chi_l, \zeta_l$ die Variabeln $u_1, \cdots, u_n$ durch Auflösung von (138) nach $\zeta_1, \cdots, \zeta_n$ einzutragen sind. Die Q_k sind dann offenbar Potenzreihen in $u_1, \cdots, u_n$ ohne konstante und lineare Glieder. Nun wollen wir die Koeffizienten der P_k so zu bestimmen versuchen, dass in keiner der Reihen Q_k Potenzprodukte der p Variabeln $u_1, \cdots, u_p$ allein auftreten; es soll also identisch

$$Q_k(u_1, \cdots, u_p, 0, \cdots, 0) = 0 \qquad (k = 1, \cdots, n) \tag{140}$$

gelten.

Nach (138) sind $\zeta_1, \cdots, \zeta_p$ Potenzreihen der p Variabeln $u_1, \cdots, u_p$ allein, und für $u_{p+1} = 0, \cdots, u_n = 0$ ist ausserdem

$$\zeta_k = P_k(\zeta_1, \cdots, \zeta_p) \qquad (k = p + 1, \cdots, n).$$

Daher sind die Bedingungen (140) gleichbedeutend mit

$$\begin{aligned} &\chi_k(\zeta_1, \cdots, \zeta_p, P_{p+1}, \cdots, P_n) + \lambda_k P_k \\ &\quad - \sum_{l=1}^{p} \frac{\partial P_k}{\partial \zeta_l} \{\chi_l(\zeta_1, \cdots, \zeta_p, P_{p+1}, \cdots, P_n) + \lambda_l \zeta_l\} = 0 \end{aligned} \tag{141}$$

für $k = 1, \cdots, n$, identisch in $\zeta_1, \cdots, \zeta_p$. Wir zerlegen

$$P_k = \sum_{q=2}^{\infty} P_{kq} \qquad (k = 1, \cdots, n),$$

wo P_{kq} ein homogenes Polynom q^{ten} Grades in $\zeta_1, \cdots, \zeta_p$ bedeutet; entsprechend sei R_{kq} der homogene Bestandteil q^{ten} Grades in $\zeta_1, \cdots, \zeta_p$ auf der linken Seite

von (141). Dann gilt offenbar

$$R_{kq} = \lambda_k P_{kq} - \sum_{l=1}^{p} \frac{\partial P_{kq}}{\partial \zeta_l} \lambda_l \zeta_l + H_{kq}, \tag{142}$$

wo H_{kq} ein Polynom in $\zeta_1, \cdots, \zeta_p, P_{lh}$ $(l = p+1, \cdots, n; h = 2, \cdots, q-1)$ und $\dfrac{\partial P_{kh}}{\partial \zeta_l}$ $(l = 1, \cdots, p; h = 2, \cdots, q-1)$ ist. Es seien bereits alle P_{lh} für $l = 1, \cdots, n$ und $h = 2, \cdots, q-1$ bekannt; diese Annahme ist inhaltslos für $q = 2$. Dann haben wir P_{kq} $(k = 1, \cdots, n)$ zufolge (141), (142) aus den Bedingungen

$$-\lambda_k P_{kq} + \sum_{l=1}^{p} \frac{\partial P_{kq}}{\partial \zeta_l} \lambda_l \zeta_l = H_{kq} \tag{143}$$

zu ermitteln, deren rechte Seiten bekannt sind.

Ist $c\zeta_1^{g_1} \cdots \zeta_p^{g_p}$ ein Summand von H_{kq}, so ergibt sich für den Koeffizienten σ des entsprechenden Gliedes von P_{kq} aus (143) die lineare Gleichung

$$\left(-\lambda_k + \sum_{l=1}^{p} g_l \lambda_l\right)\sigma = c,$$

und danach erhält man σ, und zwar eindeutig, wenn

$$\sum_{l=1}^{p} g_l \lambda_l \neq \lambda_k \tag{144}$$

ist. Da $g_1, \cdots, g_p$ nicht-negative ganze Zahlen mit der Summe $q \geqq 2$ und die reellen Teile von $\lambda_1, \cdots, \lambda_p$ negativ, die von $\lambda_{p+1}, \cdots, \lambda_n$ positiv sind, so ist (144) für $k > p$ stets erfüllt und für $k = 1, \cdots, p$ jedenfalls bei genügend grossem q. Wir setzen zunächst voraus, dass (144) ausnahmslos erfüllt ist, für $k = 1, \cdots, n$ und alle Systeme nicht-negativer ganzer Zahlen $g_1, \cdots, g_p$, deren Summe grösser als 1 ist. Dann sind also die Potenzreihen P_k auf genau eine Weise so bestimmbar, dass (140) erfüllt wird.

Es muss nun untersucht werden, wie es mit der Konvergenz der formal gefundenen Reihen P_k bestellt ist. Diese Untersuchung erfolgt am bequemsten mit der Cauchyschen Majorantenmethode. Sind P und Q zwei Potenzreihen in den gleichen Variabeln, so soll das Zeichen

$$P \prec Q$$

bedeuten, dass für je zwei entsprechende Koeffizienten α und β von P und Q die Ungleichung

$$|\alpha| \leqq \beta$$

gilt. Nach Voraussetzung sind die Potenzreihen ψ_k in (133), also auch die Reihen χ_k in (137) konvergent in einer gewissen Umgebung des Nullpunktes. Bedeutet C_1, wie auch weiterhin $C_2, \cdots, C_6$ eine geeignete positive Konstante,

so ist also

$$\chi_k \prec \frac{C_1(\zeta_1 + \cdots + \zeta_n)^2}{1 - C_1(\zeta_1 + \cdots + \zeta_n)} = f(\zeta_1, \cdots, \zeta_n) \quad (k = 1, \cdots, n),$$

wo die rechte Seite durch ihre Potenzreihe zu ersetzen ist.

Wegen (144) ist nun

$$1 + g_1 + \cdots + g_p < C_2 \left| -\lambda_k + \sum_{l=1}^{p} g_l \lambda_l \right|, \tag{145}$$

für $k = 1, \cdots, n$ und jedes System nicht-negativer ganzer Zahlen $g_1, \cdots, g_p$, deren Summe mindestens gleich 2 ist. Wir ersetzen (141) durch die abgeänderten Gleichungen

$$P_k^* + \sum_{l=1}^{p} \frac{\partial P_k^*}{\partial \zeta_l} \zeta_l = C_2 \left(1 + \sum_{l=1}^{p} \frac{\partial P_k^*}{\partial \zeta_l}\right) f(\zeta_1, \cdots, \zeta_p, P_{p+1}^*, \cdots, P_n^*) \qquad (k = 1, \cdots, n) \tag{146}$$

für n unbekannte Potenzreihen $P_k^*(\zeta_1, \cdots, \zeta_p)$. Diese lassen sich durch dasselbe rekursive Verfahren bestimmen wie die P_k, und man entnimmt durch Vergleich von (141) und (146) wegen (145) unmittelbar die Beziehung

$$P_k \prec P_k^* \qquad (k = 1, \cdots, n). \tag{147}$$

Aus (146) ersieht man, dass alle P_k^* $(k = 1, \cdots, n)$ identisch gleich sind; es sei P^* der gemeinsame Wert. Setzt man noch $\zeta_1 = \zeta_2 = \cdots = \zeta_p = \zeta$, so möge P^* in $P(\zeta)$ übergehen; ferner sei

$$f(\zeta, P) = \frac{C_1\{p\zeta + (n-p)P\}^2}{1 - C_1\{p\zeta + (n-p)P\}}.$$

Offenbar ist dann

$$P^*(\zeta_1, \cdots, \zeta_p) \prec P(\zeta_1 + \cdots + \zeta_p) \tag{148}$$

und nach (146)

$$P + \zeta \frac{dP}{d\zeta} = C_2 \left(1 + \frac{dP}{d\zeta}\right) f(\zeta, P).$$

Durch Einsetzen der Potenzreihe

$$P(\zeta) = \sum_{q=2}^{\infty} d_q \zeta^q$$

erhält man

$$\sum_{q=2}^{\infty} (q+1)\, d_q \zeta^q = C_2 \left(1 + \sum_{q=2}^{\infty} q\, d_q \zeta^{q-1}\right) f(\zeta, P).$$

Die hieraus entstehenden Rekursionsformeln für die Koeffizienten d_q zeigen,

dass

$$P < \zeta M_1 \tag{149}$$

ist, wo M_1 der kubischen Gleichung

$$\zeta M_1 = \frac{C_3(\zeta + \zeta M_1)^2}{1 - C_3(\zeta + \zeta M_1)}\,(1 + M_1)$$

genügt. Nun ist

$$\frac{(1 + M_1)^3}{1 - C_3\zeta(1 + M_1)} < \frac{1}{1 - C_4(\zeta + M_1)}, \qquad C_3\zeta < M_1$$

und folglich

$$M_1 < M_2 \tag{150}$$

mit

$$M_2 = \frac{C_5\zeta}{1 - C_5 M_2}.$$

Aus der Gleichung

$$(1 - 2C_5M_2)^{-2} = (1 - 4C_5^2\zeta)^{-1}$$

erhält man endlich

$$M_2 < M \tag{151}$$

für

$$1 + 4C_5M = (1 - 4C_4^2\zeta)^{-1},$$

$$M = C_5\zeta(1 - 4C_5^2\zeta)^{-1},$$

also nach (147), (148), (149), (150), (151)

$$P_k < \frac{C_6(\zeta_1 + \cdots + \zeta_p)^2}{1 - C_6(\zeta_1 + \cdots + \zeta_p)} \qquad (k = 1, \cdots, n).$$

Damit ist die Konvergenz der Potenzreihen P_k für hinreichend kleine Werte der absoluten Beträge von $\zeta_1, \cdots, \zeta_p$ bewiesen.

Schliesslich sind noch die Realitätsverhältnisse zu untersuchen. Aus der Lösung $P_1, \cdots, P_n$ von (141) entsteht eine weitere Lösung, indem man für jede reelle Wurzel λ_k die Grösse P_k durch $\overline{P}_k$ ersetzt und für je zwei konjugiert komplexe Wurzeln λ_j, λ_k das Paar P_j, P_k durch $\overline{P}_k, \overline{P}_j$. Da aber die Lösung eindeutig festgelegt ist, so gilt $P_k = \overline{P}_k$ im ersten Falle und $P_k = \overline{P}_j$ im zweiten Falle. Nach (138) ist also u_k reell für reelles λ_k und $u_k = \bar{u}_j$ für $\lambda_k = \bar{\lambda}_j$.

Wir wollen nunmehr sämtliche Lösungen der Differentialgleichungen (133) bestimmen, die für $s \to \infty$ in den Nullpunkt einmünden. Nach (136) und (138) genügt es, diese Untersuchung an dem System (139) vorzunehmen. Da s nicht explizit in den Differentialgleichungen auftritt, so kann man voraussetzen, dass

die gesuchten Lösungen für alle $s \geqq 0$ der Ungleichung

$$\sum_{k=1}^{n} |u_k|^2 < \epsilon \tag{152}$$

genügen, wo ϵ eine hinreichend klein zu wählende positive Konstante bedeutet. Setzt man

$$\sum_{k=p+1}^{n} |u_k|^2 = Z, \tag{153}$$

so gilt zufolge (139)

$$\frac{dZ}{ds} = \sum_{k=p+1}^{n} (\lambda_k + \bar{\lambda}_k) u_k \bar{u}_k + \sum_{k=p+1}^{n} (u_k \bar{Q}_k + \bar{u}_k Q_k). \tag{154}$$

Nach (140) ist nun jedes Glied der Potenzreihen Q_k und $\bar{Q}_k$ durch eine der Variabeln $u_{p+1}, \cdots, u_n$ teilbar, also jedes Glied der Potenzreihe für den zweiten Summanden auf der rechten Seite von (154) durch ein Produkt zweier dieser Variabeln teilbar. Da diese Potenzreihe mit Gliedern dritter Ordnung beginnt, so ist nach (152) und (153) für genügend kleines ϵ die Ungleichung

$$\sum_{k=p+1}^{n} (u_k \bar{Q}_k + \bar{u}_k Q_k) \geqq -\rho_n Z \tag{155}$$

erfüllt, also nach (134), (153), (154)

$$\frac{dZ}{ds} \geqq 2 \sum_{k=p+1}^{n} \rho_k |u_k|^2 - \rho_n Z \geqq \rho_n Z,$$

$$\frac{d(Ze^{-\rho_n s})}{ds} \geqq 0.$$

Daher ist der Ausdruckt $Ze^{-\rho_n s}$ für alle $s \geqq 0$ monoton wachsend; andererseits strebt er für $s \to \infty$ nach 0, weil ρ_n positiv und $Z < \epsilon$ ist. Folglich ist für die gesuchte Lösung $Z = 0$, also

$$u_k = 0 \qquad (k = p + 1, \cdots, n). \tag{156}$$

Nach (140) reduzieren sich jetzt die Differentialgleichungen (139) auf die einfache Form

$$\frac{du_k}{ds} = \lambda_k u_k \qquad (k = 1, \cdots, p) \tag{157}$$

mit der Lösung

$$u_k = \alpha_k e^{\lambda_k s} \qquad (k = 1, \cdots, p). \tag{158}$$

Dabei ist α_k reell für reelles λ_k und $\alpha_k = \bar{\alpha}_j$ für $\lambda_k = \bar{\lambda}_j$. Da die Realteile von $\lambda_1, \cdots, \lambda_p$ negativ sind, so mündet die durch (156) und (158) gegebene Lösung bei beliebigen α_k tatsächlich für $s \to \infty$ in den Nullpunkt ein.

Durch die Bedingungen (156) wird eine analytische Fläche von p Dimensionen im Raume der $\delta_1, \cdots, \delta_n$ definiert. Man erhält eine Darstellung der Fläche durch die Parameter $\zeta_1, \cdots, \zeta_p$, wenn man in die Ausdrücke

$$\delta_k = \sum_{l=1}^{n} h_{kl}\zeta_l \qquad (k = 1, \cdots, n)$$

für $\zeta_{p+1}, \cdots, \zeta_n$ die aus (138), (156) folgenden Werte

$$\zeta_k = P_k(\zeta_1, \cdots, \zeta_p) \qquad (k = p + 1, \cdots, n)$$

einträgt. Will man eine reelle Parameterdarstellung haben, so hat man das Paar ζ_j, ζ_k für $\lambda_k = \bar{\lambda}_j$ durch $(\zeta_j + \zeta_k)/2$, $(\zeta_j - \zeta_k)/2i$ zu ersetzen. Die gesuchten Bahnkurven erfüllen dann genau diese p-dimensionale Fläche, und zwar erhält man die einzelnen Lösungen, indem man vermöge der Gleichungen

$$u_k = \zeta_k - P_k(\zeta_1, \cdots, \zeta_p) \qquad (k = 1, \cdots, p)$$

die Grössen $\zeta_1, \cdots, \zeta_p$ als Potenzreihen von $u_1, \cdots, u_p$ bestimmt und dann

$$u_k = \alpha_k e^{\lambda_k s}$$

setzt, mit beliebigen Konstanten α_k $(k = 1, \cdots, p)$ unter Beachtung der Realitätsbedingungen. Die allgemeine Lösung hängt also von p reellen Parametern ab.

Es seien noch einmal die beiden Voraussetzungen erwähnt, die wir in diesem Paragraphen an verschiedenen Stellen der Untersuchung eingeführt haben, nämlich die Verschiedenheit der Wurzeln $\lambda_1, \cdots, \lambda_n$ und das Bestehen der Ungleichung (144). Wir wollen noch feststellen, in welcher Art unsere Ergebnisse zu modifizieren sind, wenn wir diese Voraussetzungen fallen lassen.

12. Ausartungen

Wir verzichten nun auf die Annahme (144), halten aber zunächst noch an der Voraussetzung fest, dass $\lambda_1, \cdots, \lambda_n$ verschieden sind. Wir denken uns dann die endlich vielen Lösungssysteme der diophantischen Gleichung

$$\sum_{l=1}^{p} g_l\lambda_l = \lambda_k \tag{159}$$

bestimmt, wobei k irgend ein Index der Reihe $1, \cdots, p$ ist und $g_1, \cdots, g_p$ nicht-negative ganze Zahlen, deren Summe mindestens 2 beträgt. In dem Ansatz (138) waren bisher $P_1, \cdots, P_n$ Potenzreihen in $\zeta_1, \cdots, \zeta_p$ ohne konstante und lineare Glieder, deren Koeffizienten rekursiv eindeutig durch die Bedingung (140) festgelegt wurden. Jetzt schliessen wir aus dem mit unbestimmten Koeffizienten gebildeten Potenzreihen $P_1, \cdots, P_n$ sämtliche Glieder $\sigma\zeta_1^{g_1} \cdots \zeta_p^{g_p}$ aus, deren Exponenten einer Gleichung (159) genügen, und ersetzen (140) durch die folgende abgeschwächte Bedingung: Für $k = 1, \cdots, p$ soll eine Identität

$$Q_k(u_1, \cdots, u_p, 0, \cdots, 0) = V_k(u_1, \cdots, u_p) \tag{160}$$

gelten, wo V_k ein Polynom aus solchen Gliedern $cu_1^{g_1} \cdots u_p^{g_p}$ bedeutet, deren

Exponenten (159) erfüllen; für $k = p + 1, \cdots, n$ soll wie bisher

$$Q_k(u_1, \cdots, u_p, 0, \cdots, 0) = 0 \tag{161}$$

gelten. Es lässt sich auf dem früheren Wege ohne Mühe zeigen, dass die Potenzreihen $P_1, \cdots, P_n$ wieder eindeutig bestimmt und konvergent sind, und auch die Polynome $V_1, \cdots, V_p$ sind eindeutig fixiert. Da zur Herleitung der Ungleichung (155) die Formeln (140) nur für $k = p + 1, \cdots, n$ herangezogen werden, so bleibt die zu (156) führende Schlussweise bestehen. Bei sämtlichen für $s \to \infty$ nach dem Nullpunkt wandernden Lösungen von (133) ist also wieder $u_{p+1} = 0, \cdots, u_n = 0$. An die Stelle des Systems (157) tritt aber jetzt nach (139) und (160)

$$\frac{du_k}{ds} = \lambda_k u_k + V_k(u_1, \cdots, u_p) \qquad (k = 1, \cdots, p). \tag{162}$$

Nach (134) ist für jede Lösung von (159)

$$g_k = 0, \qquad g_{k+1} = 0, \cdots, g_p = 0,$$

folglich ist $V_k = V_k(u_1, \cdots, u_{k-1})$ ein Polynom in $u_1, \cdots, u_{k-1}$ allein und $V_1 = 0$. Die Integration von (162) lässt sich ohne weiteres ausführen und ergibt

$$\begin{aligned} u_1 &= \alpha_1 e^{\lambda_1 s}, \\ u_2 &= \alpha_2 e^{\lambda_2 s} + sV_2(u_1) = \{\alpha_2 + sV_2(\alpha_1)\}e^{\lambda_2 s}, \end{aligned} \tag{163}$$

allgemein durch vollständige Induktion

$$u_k = (\alpha_k + W_k)e^{\lambda_k s} \qquad (k = 1, \cdots, p), \tag{164}$$

wo W_k ein eindeutig bestimmtes Polynom in $\alpha_1, \cdots, \alpha_{k-1}$ und s ist. Da umgekehrt bei beliebiger Wahl der Konstanten $\alpha_1, \cdots, \alpha_p$ die durch (164) gegebenen Funktionen u_k für $s \to \infty$ gegen 0 streben, so gelten die Ergebnisse des vorangehenden Paragraphen auch in diesem Fall, wenn nur u_k durch (164) erklärt wird.

Endlich lassen wir auch noch die Voraussetzung fallen, dass die Wurzeln $\lambda_1, \cdots, \lambda_n$ alle verschieden sind. Dann ist die Matrix $\mathfrak{Q} = (q_{kl})$ in (135) im allgemeinen keine reine Diagonalmatrix: Es ist wieder $q_{kk} = \lambda_k$, $q_{kl} = 0$ für $k \neq l$ und $k \neq l + 1$, $q_{kl} = 0$ für $k = l + 1$ und $\lambda_l \neq \lambda_{l+1}$; für $k = l + 1$ und $\lambda_l = \lambda_{l+1}$ ist dagegen q_{kl} entweder gleich 0 oder gleich λ_l. An die Stelle des Systems (137) tritt jetzt allgemeiner

$$\frac{d\zeta_1}{ds} = \lambda_1\zeta_1 + \chi_1,$$

$$\frac{d\zeta_k}{ds} = \lambda_k(\zeta_k + e_k\zeta_{k-1}) + \chi_k \qquad (k = 2, \cdots, n);$$

dabei ist $e_k = 0$ für $\lambda_k \neq \lambda_{k-1}$, während im Falle $\lambda_k = \lambda_{k-1}$ entweder $e_k = 0$ oder $e_k = 1$ ist. Anstatt (135) erhält man durch die früher benutzte Methode

die Differentialgleichungen

$$
(165)\qquad \begin{cases} \dfrac{du_1}{ds} = \lambda_1 u_1 + Q_1(u_1, \cdots, u_n), \\[2ex] \dfrac{du_k}{ds} = \lambda_k(u_k + e_k u_{k-1}) + Q_k(u_1, \cdots, u_n) \qquad (k = 2, \cdots, n), \end{cases}
$$

wo $Q_1, \cdots, Q_n$ wieder den Bedingungen (160) und (161) genügen, mit der dort angegebenen Bedeutung der Polynome $V_1, \cdots, V_p$. Anstelle des in (153) erklärten Ausdrucks Z betrachten wir allgemeiner

$$
\sum_{k=p+1}^{n} h_k \,|\, u_k \,|^2 = Z_0
$$

mit konstanten Werten $h_{p+1}, \cdots, h_n$. Zufolge (165) gilt dann

$$
(166)\qquad \frac{dZ_0}{ds} = \sum_{k=p+1}^{n} h_k\{(\lambda_k + \bar\lambda_k) \,|\, u_k \,|^2 + e_k(\lambda_k u_{k-1} \bar u_k + \bar\lambda_k \bar u_{k-1} u_k)\} + \sum_{k=p+1}^{n} (u_k \bar Q_k + \bar u_k Q_k).
$$

Wegen $\lambda_p \neq \lambda_{p+1}$ ist nun $e_{p+1} = 0$, und der erste Summand auf der rechten Seite von (166) lässt sich in der Form

$$
H = \rho_n h_n \,|\, u_n \,|^2 + \sum_{k=p+2}^{n} \left(\rho_{k-1} h_{k-1} - e_k \frac{|\,\lambda_k\,|^2}{\rho_k} h_k\right) |\, u_{k-1} \,|^2 + \sum_{k=p+1}^{n} \rho_k h_k \left| u_k + e_k \frac{\lambda_k}{\rho_k} u_{k-1} \right|^2
$$

schreiben. Wählt man speziell

$$
h_{p+1} = 1, \qquad h_{k+1} = \frac{\rho_k \rho_{k+1}}{2 \,|\, \lambda_{k+1} \,|^2} h_k \qquad (k = p+1, \cdots, n-1),
$$

so wird offenbar

$$
(167)\qquad H \geqq \tfrac{1}{2} \sum_{k=p+1}^{n} \rho_k h_k \,|\, u_k \,|^2 \geqq \tfrac{1}{2} \rho_n Z_0 .
$$

Aus (166) und (167) folgt jetzt, wenn ϵ in (152) genügend klein gewählt wird, die Ungleichung

$$
\frac{dZ_0}{ds} \geqq \tfrac{1}{3} \rho_n Z_0
$$

und daraus wieder das Verschwinden von $u_{p+1}, \cdots, u_n$ bei sämtlichen Lösungen von (165), die für $s \to \infty$ im Nullpunkt einmünden. Es bleibt noch das System

$$
\frac{du_1}{ds} = \lambda_1 u_1,
$$

$$
\frac{du_k}{ds} = \lambda_k(u_k + e_k u_{k-1}) + V_k(u_1, \cdots, u_{k-1}) \qquad (k = 2, \cdots, p)
$$

zu integrieren. Die Lösung hat wieder die Gestalt

$$u_k = (\alpha_k + W_k)e^{\lambda_k s} \qquad (k = 1, \cdots, p),$$

wo W_k ein eindeutig bestimmtes Polynom in s und den Integrationskonstanten $\alpha_1, \cdots, \alpha_{k-1}$ bedeutet; insbesondere ist

$$W_1 = 0, \qquad W_2 = s\{e_2\lambda_2\alpha_1 + V_2(\alpha_1)\}.$$

Damit haben wir die Resultate des vorigen Paragraphen auf den Fall mehrfacher Wurzeln $\lambda_1, \cdots, \lambda_n$ übertragen, abgesehen von der Untersuchung der Realitätsverhältnisse. Diese Untersuchung liesse sich ohne Schwierigkeit durchführen; wir gehen aber darauf nicht mehr ein, da bei der Anwendung auf die Dreierstosslösungen im Falle einer mehrfachen Wurzel alle Wurzeln reell sind und dann überhaupt keine imaginären Grössen in die Rechnung eingeführt zu werden brauchen.

13. Die Dreierstossbahnen

Wir wenden nunmehr die Ergebnisse der beiden vorangehenden Paragraphen auf die Untersuchung des Systemes (118) an. Die charakteristischen Wurzeln der Matrix $\mathfrak{A}$ werden im gleichseitigen Falle durch die 6 Zahlen $-a_0, -a_1, -a_2, a_3, a_4, a_5$ in (131) geliefert, im geradlinigen Falle durch die Zahlen $-b_0, -b_1, b_2, b_3, b_4, b_5$ in (132). Es sind $-a_0, -a_1, -a_2, -b_0, -b_1$ negativ und die übrigen Wurzeln sind positiv oder haben positiven Realteil. Folglich ist $p = 3$ im gleichseitigen Falle und $p = 2$ im geradlinigen Falle.

Die Wurzeln $-a_0, -a_1, -a_2$ sind sämtlich verschieden, ausser wenn $m_1 = m_2 = m_3$ ist, und dann ist $-a_1 = -a_2 = (1 - \sqrt{13})/6 > -a_0$. Die Wurzeln $-b_0, -b_1$ sind stets voneinander verschieden.

Es ist noch festzustellen, wann (159) lösbar ist und welches dann die Lösungen sind. Im gleichseitigen Falle ist

$$a_0 > a_1 \geqq a_2, \qquad 2a_1 > a_0, \qquad a_1 + a_2 > a_0;$$

gilt also

$$a_0x + a_1y + a_2z = a_k, \qquad x + y + z \geqq 2$$

für ein k der Reihe 1, 2, 3 in nicht-negativen ganzen Zahlen x, y, z, so folgt $x = 0, y = 0$, und es ist entweder

$$a_0 = ga_2, \qquad z = g \tag{168}$$

mit ganzem g oder

$$a_1 = ha_2, \qquad z = h \tag{169}$$

mit ganzem h. Die Annahme (168) ergibt nach (131) für die in (128) definierte Grösse

$$a = \frac{m_2m_3 + m_3m_1 + m_1m_2}{(m_1 + m_2 + m_3)^2}$$

den Wert

$$a = \tfrac{4}{27} g^{-4}(g+2)(3g^2 - g - 2) \qquad (g = 2, 3, \ldots), \tag{170}$$

während aus (169)

$$a = \tfrac{4}{3} h(h^2+1)^{-2}(hv+1)(h-v) \qquad (h = 1, 2, \cdots) \tag{171}$$

mit

$$v = \tfrac{1}{12}(h^2-1)(h^2+1)^{-1}\{-1 + [1 + 24(h^2+1)(h+1)^{-2}]^{\frac{1}{2}}\}$$

folgt. Hieraus ist noch leicht ersichtlich, dass nicht (168) und (169) zugleich eintreten können. Beim geradlinigen Falle ist $b_1 > b_0$; gilt also

$$b_0 x + b_1 y = b_k, \qquad x + y \geqq 2$$

für ein k der Reihe 1, 2 in nicht-negativen ganzen Zahlen x, y, so ist

$$b_1 = jb_0, \qquad x = j, \qquad y = 0$$

mit ganzem j, und die in (130) erklärte Grösse

$$b = \frac{m_1\{1 + (1-\omega)^{-1} + (1-\omega)^{-2}\} + m_3(1 + \omega^{-1} + \omega^{-2})}{m_1 + m_2\{\omega^{-2} + (1-\omega)^{-2}\} + m_3}$$

hat nach (132) den Wert

$$b = j^2 + \tfrac{1}{2}(j-3). \tag{172}$$

Im gleichseitigen Falle bilden die für $s \to \infty$ in den Nullpunkt einmündenden Lösungen von (118) eine dreidimensionale analytische Mannigfaltigkeit. Ist nicht (170) oder (171) erfüllt, so lassen sich $\delta_1, \cdots, \delta_6$ in Potenzreihen der Variabeln

$$u_1 = \alpha_1 e^{-a_0 s}, \qquad u_2 = \alpha_2 e^{-a_1 s}, \qquad u_3 = \alpha_3 e^{-a_2 s} \tag{173}$$

entwickeln; dabei hängen die Koeffizienten der Potenzreihen nur von m_1, m_2, m_3 ab und α_1, α_2, α_3 sind willkürliche reelle Konstanten. Ist (170) erfüllt, so hat man auf Grund von (163)

$$u_1 = (\alpha_1 - c_1 \alpha_3^g s)e^{-a_0 s} \tag{174}$$

zu setzen, wo c_1 eine von m_1, m_2, m_3 abhängige Konstante bedeutet. Im Falle (171) ist entsprechend

$$u_2 = (\alpha_2 - c_2 \alpha_3^h s)e^{-a_1 s}. \tag{175}$$

Im geradlinigen Falle bilden die für $s \to \infty$ in den Nullpunkt einmündenden Lösungen von (118) eine zweidimensionale analytische Mannigfaltigkeit. Ist nicht (172) erfüllt, so lassen sich $\delta_1, \cdots, \delta_6$ in Potenzreihen der Variabeln

$$v_1 = \beta_1 e^{-b_0 s}, \qquad v_2 = \beta_2 e^{-b_1 s}$$

entwickeln, wo die Koeffizienten nur von m_1, m_2, m_3 abhängen und β_1, β_2

willkürliche reelle Konstanten bedeuten. Ist (172) erfüllt, so hat man

$$v_2 = (\beta_2 - c_3\beta_1^j s)e^{-b_1 s}$$

zu setzen.

Trägt man die gefundenen Potenzreihen für $\delta_1, \cdots, \delta_8$ in die rechte Seite von (119) ein, so erhält man $\delta_8 = p_4$ durch eine einfache Quadratur. Da das unbestimmte Integral einer Potenzreihe in den durch (173) oder (174), (175) definierten Grössen u_1, u_2, u_3 wieder eine solche Potenzreihe ist und das gleiche für die Potenzreihen in v_1, v_2 gilt, so ist also

$$p_4 = \gamma + P,$$

wo γ eine willkürliche Konstante und P eine eindeutig festgelegte Potenzreihe in u_1, u_2, u_3 oder in v_1, v_2 ohne konstantes Glied ist. Folglich hat p_4 einen Grenzwert für $s \to \infty$, nämlich die Zahl γ. Damit ist also endlich bewiesen, dass die 3 Massenpunkte im ruhenden Koordinatensystem in bestimmten Richtungen zusammenstossen. Durch eine geeignete Drehung des Koordinatensystems kann man erreichen, dass $\gamma = 0$ ist.

Trägt man die gefundenen Reihen nach (96), (109), (114) in die Werte der Koordinaten x_k, y_k $(k = 1, 2, 3)$ ein und benutzt (110), so erhält man im gleichseitigen Falle für jede Koordinate eine Darstellung

$$w = t^{2/3} w^*(u_1, u_2, u_3),$$

wo $w^*(u_1, u_2, u_3)$ eine Reihe nach positiven Potenzen von

$$u_1 = \alpha_1 t^{2/3}, \qquad u_2 = \alpha_2 t^{a_1}, \qquad u_3 = \alpha_3 t^{a_2}$$

bedeutet, deren Koeffizienten nur von m_1, m_2, m_3 abhängen. Liegt eine der Ausartungen (170) oder (171) vor, so ist statt dessen

$$u_1 = (\alpha_1 + c_1\alpha_3^g \log t)t^{2/3}$$

oder

$$u_2 = (\alpha_2 + c_2\alpha_3^h \log t)t^{a_1}$$

zu setzen. Im geradlinigen Falle hat man analog

$$w = t^{2/3} w^*(v_1, v_2)$$

mit

$$v_1 = \beta_1 t^{2/3}, \qquad v_2 = \beta_2 t^{b_1}$$

oder, wenn (172) erfüllt ist, mit

$$v_2 = (\beta_2 + c_3\beta_1^j \log t)t^{b_1}.$$

Schliesslich kann man noch eine Drehung des Koordinatensystems um konstante Winkel vornehmen und ihm eine Translation mit konstanter Geschwindigkeit erteilen.

Die Werte a_1, a_2 und b_1 sind irrational, wenn m_1, m_2, m_3 nicht im Bereich der rationalen Zahlen in gewisser Weise algebraisch abhängig voneinander sind. Daher ist der Punkt $t = 0$ im allgemeinen ein logarithmischer Verzweigungspunkt für die Koordinaten der Dreierstossbahnen.

[Zusatz bei der Korrektur:] Inzwischen bin ich aufmerksam geworden auf eine Arbeit von G. Sokoloff,[6] welche ebenfalls der Untersuchung des Dreierstosses gewidmet ist und Reihenentwicklungen für die Koordinaten der kollidierenden Massenpunkte enthält. Im Beweis findet sich eine Lücke, die aber durch Benutzung des Bohlschen Satzes leicht ausgefüllt werden kann. Die vorliegende Darstellung dürfte in wesentlichen Punkten den Vorzug grösser Einfachheit haben.

PRINCETON, N. J.

[6] G. Sokoloff, *Conditions d'une collision générale des trois corps qui s'attirent mutuellement suivant la loi de Newton*, Académie des Sciences de l'Ukraine, Mémoires de la classe des sciences physiques et mathématiques, Bd. 9 (1928), S. 1–64.

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INTUITIVE PROBABILITIES AND SEQUENCES[1]

BY B. O. KOOPMAN

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This paper forms a second part of the general study of probability regarded as a branch of intuitive logic, the first part having appeared in *The Axioms and Algebra of Intuitive Probability*.[2] The point of view, notation, assumptions and theorems of the latter are assumed in the present work. The chief object here is to set forth the connection between the objective notion of statistical weight or frequency in a sequence and the intuitive conceptions of our theory.

It is necessary first to examine the class of definitions of probability based on the notion of frequency in a sequence (collectives); we shall reach the conclusion that *every application of such definitions of probability to experimental science implicitly presupposes the a priori intuitive conception of probability* (§1). Further progress in the exact formulation of the ideas requires the scrutiny of certain logical discriminations involved in the relation of *asserted* and *contemplated* propositions, a discrimination the ignoring of which leads to fallacies (§2). The notion of *symmetry* and the related one of *independence* in a set of trials is then studied (§3); the former replaces in the present theory of sequences the notion of *random* in the usual theory of collectives. Finally the theorems relating statistical weight with intuitive probability (numerical and otherwise) are established.

There is thus achieved a complete synthesis of the intuitive concept of probability with the objective notion of statistical weight which forms the basis of statistics and quantum mechanics,—and this without the assumption of a single general principle beyond those which have already been posited in our earlier work.

1. Probability as a Frequency

An event E (such as the tossing of a coin, the search for a photon with a sensitive plate, the observation of fatality following an inoculation, etc.) is regarded as having two possible outcomes, "success" (labeled 1) and "failure" (labeled 0). An infinite sequence of trials of E are made (conceptually) under essentially the same conditions, the results (1 or 0) forming the sequence (α): $(\alpha_1, \alpha_2, \cdots)$ $(\alpha_i = 0 \text{ or } 1)$. The theory of probability in collectives starts with the following assumptions.[3]

[1] Presented to the American Mathematical Society April 26, 1940.

[2] Annals of Math., Vol. 41, No. 2 (1940). Presented to the American Mathematical Society February 25, 1939. This paper will be referred to herein as AAP.

[3] Cf., e.g., R. von Mises, *Probability, Statistics and Truth* (London 1939). A. H. Copeland, *Consistency of the Conditions Determining Kollektivs*, Trans. Amer. Math. Soc., Vol. 42 (1937) p. 333, as well as the references contained therein.

First assumption. $\lim_{n\to\infty}(\alpha_1 + \cdots + \alpha_n)/n = w$.

Second assumption. If (α'):$(\alpha_1', \alpha_2', \cdots)$ *is a sub-sequence of* (α) *defined by any place-selection s of the class S then* $\lim_{n\to\infty}(\alpha_1' + \cdots + \alpha_n')/n = w$.

Here a place-selection is regarded as defined by a precise mathematical rule, and the class S as having been exactly delimited: we shall raise no question on this point.[4] Any sequence (α) for which these assumptions are made is called a *collective* (properly: *with respect to* S), and according to what shall be termed herein the collectivist definition of probability, *the number* w (naturally $0 \leqq w \leqq 1$) *is defined as the probability of success of* E (*in the collective* (α)).

Our object is not with the mathematical theory of collectives. We wish merely to ascertain whether the collectivist definition of probability can be applied to an experimental phenomenon without an auxiliary notion of an *a priori* subjective order reducing in essence to that of intuitive probability. And since we are dealing only with Assumption 1, our critique will apply to the whole class of theories which regard probability as merely a frequency in a sequence.

Let us suppose that I am given the information that (α) is a collective, that I know the definition of S, and also the value of w (e.g., let $w = 0.1$). Furthermore, let me excise from my brain all *a priori* intuitive notions of "likelihood", "degree of conviction", "reasonableness", etc., so that only my strict logical and observational powers remain. Just what conclusion can I derive from this knowledge concerning (α) and applying to the results of the totality of trials of E which I can make in my lifetime? Since I can only make a number N of trials obviously $<10^{10}$, and inasmuch as any set of values $(\alpha_1, \cdots, \alpha_N)$ ($\alpha_n = 0$ or 1) will be mathematically consistent with the assumptions regarding (α), it follows that I can draw absolutely no conclusion at all (in the only terms which I can understand, i.e., strict logic), so my information regarding (α) is totally irrelevant to my experience with E.

The situation is exactly the same for any individual whose name we can give. Even if it could be proved that some individual exists for whose set of trials of E surely $(\alpha_1 + \cdots + \alpha_N)/N = 0.1$, we could never inform him that he was this favored one, nor could he ever know it himself—until, indeed, his trials of E are all completed, in which case the information concerning (α) would become unnecessary. And an individual can be replaced by all society during a given historical era: The number N of total possible trials which it can make is much larger, but once the era has been named an upper limit of N is set and any values of $(\alpha_1, \cdots, \alpha_N)$ are consistent with the assumptions concerning (α). It may be said that a time will come when all observations made thereunto will give for $(\alpha_1 + \cdots + \alpha_N)/N$ a value close to w. But this does not remove the difficulty: The time will never come when it can be *proved* that the proper value of N has been reached.

A finite theory of collectives has been proposed in which the finite set

[4] For a discussion from the point of view of modern logic cf. A. Church, *On the Concept of a Random Sequence*, Bull. Amer. Math. Soc. Vol. 46 (1940) p. 130.

$(\alpha_1, \cdots, \alpha_M)$ replaces the infinite sequence $(\alpha_1, \alpha_2, \cdots)$ and such ratios as $(\alpha_1 + \cdots + \alpha_N)/N$ replace the corresponding limits.[5] Here M is regarded as enormous—"practically infinite"—so that the previous N is minute in comparison, and any values of $(\alpha_1, \cdots, \alpha_N)$ are, here also, consistent with any values of the above ratio within limits of experimental error: The earlier difficulty is encountered here as before. This will naturally be quite otherwise if N is of the same order of magnitude as M; but in this case (even assuming that the experimenter could ever know it) the theory will be superfluous. The matter can be put thus: The problem of sampling is insoluble—nay, unstable—in terms of strict logic divorced from all intuitive notions of probability.

Certain philosophers regard the theory of collectives as affording not a theory of knowledge but a pattern of behavior.[6] But it is always a question not of *any* behavior but of a *most favorable* behavior in a certain highly precise sense. And the proposition that the particular pattern of behavior derived from the theory of collectives is actually this most favorable one becomes subject to all the aforementioned difficulties.

Leaving collectives for the moment, we consider the great class of physical quantities which are defined as limits. It soon appears that here also the result of no measurement of the sort which can be carried out in the laboratory either determines or is determined by what is deemed the true value of such a quantity —by strict mathematical logic alone.[7] For example the density ρ of a continuous medium at the point P is defined as $\rho = \lim M/V$ (as $V \to P$): The finite set of ratios M/V which I can ever measure have values totally devoid of strict logical relationship with the value of ρ. Nor is the difficulty due to the idealization involved in the definition of ρ: If ρ is defined simply as the value of M/V measured by a specified person who has instruments a hundred times more effective than any which I possess, the difficulty remains the same (cf. the case of finite collectives).

Of course, when I find that my values of M/V for rapidly decreasing V vary by amounts growing negligible, the *common sense* of the experimenter leads me to the conclusion that the final will not differ appreciably from the actual value of ρ. But precisely what does it mean to conclude something by common sense which is not derivable by strict logic? If we have made the meaning of intuitive probability in the least degree clear, it must be apparent that such a conclusion of common sense is precisely an instance of a conceptualism of intuitive probability. Thus we are not entitled to it under the rigorous exclusion of all notions but those of strict logic: Our values of M/V and the value of ρ must remain without link or bind.

Returning to collectives, let us consider how those who regard probability

[5] R. von Mises, l.c., p. 121.

[6] H. Reichenbach, *Wahrscheinlichkeitslehre*, Leiden 1935.

[7] It is understood, of course, that certain quantities of the sort here considered may be derived from others by the laws of the physical theory. We merely claim that there are certain initial points at which the present difficulty arises.

as a frequency have sought to deal with the present question.[8] The limit w is regarded as of the category of (idealized) physical quantities defined by limits such as ρ; the observational material $(\alpha_1 + \cdots + \alpha_N)/N$ is the analogue of M/V. And it is regarded that conclusions applying to $(\alpha_1 + \cdots + \alpha_N)/N$ can be drawn from w and *vice versa* in exactly the same sense and with the same validity as in the case of M/V and ρ: Why should a practical experimenter hesitate to do this more in the former case than in the latter?

This reasoning of course is impeccable. But it must be clear by now that it is two-edged: either it compels us to concede that some rudimentary notion of intuitive probability is presupposed in the applications of the collectivist theory, or else it merely assimilates one insurmountable difficulty to another.

Having reached the point where it is compulsory to regard the applied notion of frequency as presupposing the intuitive concept of probability, the question is raised as to precisely how the relation between the two is to be formulated. Perhaps the most obvious way of doing this and at the same time making the fewest demands on the intuition would be to enunciate the following principle: *The intuitive probability of success of E (in (α)) increases with w.*

Put in explicitly logical form and with the intuitive symbol $(\prec)$ this becomes

Hypothesis. (i) *The sequence (α) of trials of E is a collective of frequency w.* (ii) *The sequence (α^*) of trials of a second event E^* is a collective of frequency w^*.* (iii) $w < w^*$.

Conclusion. *(E succeeds on n^{th} trial) $\prec$ (E^* succeeds on n^{th} trial).*

This principle would enable us to derive non-trivial instances of $(\prec)$ from the hypothesis (i) (ii) (iii) which do not contain any non-trivial statements involving $(\prec)$ and thus place us in the presence of a situation utterly new in our theory, and quite contrary to our view expressed at the end of §1 of AAP to the effect that no such derivation is possible in the nature of things.

It is easy to show, however, that the suggested principle must be modified.

Case 1: suppose that in a particular instance of its application we are reliably informed of the outcome of some or all of the trials of E and E^*, i.e., that we know the values of certain α_n, $\overset{*}{\alpha}_n$. Then evidently the conclusion would be absurd if we knew e.g. that $\alpha_n > \overset{*}{\alpha}_n$—whatever the values of w and w^*.

Case 2: suppose that we are informed of the results of certain of the outcomes by a person in whose reliability we place considerable but not absolute trust. If he has declared that $\alpha_n > \overset{*}{\alpha}_n$ this might lead us to deny the conclusion of the principle for the n'^{th} trial—and on perfectly reasonable grounds.

Case 3: suppose that we believe that certain more or less external events (weather, sun-spots, phases of the moon, incantations) are capable of influencing the outcome of E and E^*, being favorable let us say to E, unfavorable to E^*. Then we might well refrain from admitting the conclusion, particularly if the values of w and w^* are exceedingly close. A second person may regard our idea of an influence on E and E^* as reasonable, far-fetched, or absurd, depending on

[8] R. von Mises, l.c., p. 124.

the case in point, but he will never be able to prove that we are wrong by any reasoning or objective experiment which does not itself start by making *a priori* assumptions involving $(\prec)$—certainly not by the use of reasoning based on the suggested principle concerning sequences if a glaring *petitio* is to be avoided.

Evidently hypotheses (i) (ii) (iii) are insufficient to insure the conclusion: Something further must be added which is in the nature of an exclusion of the subsidiary knowledge (Case 1), reasonable certainty (Case 2), or disposition of belief (Case 3). Again, if we have made the conception of intuitive probability in the leastwise clear, the form which this additional hypothesis must take is evident: *It must make assertions in terms of* $(\prec)$. The precise phrasing of the condition will incorporate the idea that the various trials of E are made under essentially the same circumstances, as well as the notion that the trials are either independent events or have a constant influence upon one another. This whole question is studied in §3, the final condition being what is there termed the *symmetry* of the sequence of trials. The final addition to the hypothesis thus becomes:

HYPOTHESIS. (iv) *If* $(i_1, \cdots, i_t)$ *and* $(j_1, \cdots, j_t)$ *are any two sets each containing t distinct positive integers, then the intuitive probability that the trials of E of orders* $i_1, \cdots, i_t$ *should all lead to success is equal* $(\approx)$ *to that for trials* $j_1, \cdots, j_t$. *This is assumed for* $t = 1, 2, \cdots$.

(v) *Similarly for* E^*.

The question now arises whether under all these hypotheses (i) (ii) (iii) (iv) (v) the conclusion may reasonably be maintained, and if so, whether this fact constitutes a new principle of probability which requires to be posited. The answer is trenchant: Absolutely no new principle need be assumed, and the conclusion in question is the conclusion of a *theorem* which has as its hypothesis the assumptions (iii) (iv) (v) and the first half only of assumptions (i) and (ii). And that part of (i) and (ii) which is discarded is precisely Assumption 2 in the definition of collectives,—that very condition, namely, which has been found most unwelcome inasmuch as it involves the somewhat arbitrary choice of the class S of place-selection rules.

Before proceeding it is necessary to settle one question. Granting that $\lim (\alpha_1 + \cdots + \alpha_n)/n = w$, we may clearly rearrange the sequence (α) (at least when $0 < w < 1$) so that any other preassigned limit is obtained; does this not lead to inconsistent results? This question is answered in the negative by scrutinizing the hypothesis concerning the existence of the limit w in the light of deeper logical considerations, whereupon the hypothesis in question appears capable of two utterly distinct interpretations; the one presupposed in our theorem is not the one which permits this rearrangement. These logical preliminaries occupy us in §2.

One final remark regarding the precise formulation of this theorem in terms of the common presumption h: What right have we to assume in a given application that the concrete circumstances in the $n^{\prime\text{th}}$ trial are the same as in the $m^{\prime\text{th}}$? It would clearly be more fitting to let the presumption in the first case be h_n,

that in the second be h_m. The answer to this question is that in the case in point we assume that a certain h can be formulated for which $a_n/h_n \approx a_n/h$ $(n = 1, 2, \cdots)$: this is part of the assumption that the trials are all made "under essentially similar conditions." While it is an assumption external to the theory of probability, being a statement of the observer's attitude towards a physical situation, its precise formulation is made possible in the language of the theory, which thus fulfills exactly its appropriate function and nothing more.

These remarks have bearing on Theorems 11 and 12 below.

2. A Logical Distinction

It has been explained (AAP §2) that the present theory makes use of two distinct classes of propositions. There are firstly the *contemplated* propositions, these being the concrete statements of the outcome of particular physical or biological events verifiable in principle by the performance of appropriate crucial experiments: They are denoted by lower case Latin letters and their finite $(\sim \cdot \vee)$ combinations. There are secondly the *asserted* propositions, which can always be denoted by connecting a pair of symbols for contemplated propositions by means of a single one of the assertive symbols $(\subset, \prec)$ or their derivates $(=, \approx, <, \|,$ etc.). All the axioms and theorems of the present theory have a standard form: both hypothesis and conclusion are sets (finite or infinite) of asserted propositions; for in last analysis they are but statements of the laws of consistency governing any aggregate of logico-probability assertions which a given individual at a given moment can make.[9]

It will be convenient henceforth to depart from one convention of notation made hitherto: We will apply the logical symbols $(\sim \cdot \vee)$, as well as $\prod$ and $\sum$, to *asserted* propositions, thereby obtaining new asserted propositions *of the same logical type*. Thus for example

$$\text{(i)} \quad \sim (a/h \prec b/k); \qquad \text{(ii)} \quad (a = 1) \vee (b = 1);$$

$$\text{(iii)} \quad \prod_{n=1}^{\infty} (a_n = 1), \quad \text{i.e,} \quad (a_1 = 1)(a_2 = 1) \cdots ;$$

are regarded as mere abbreviations for assertions which may be couched respectively in the following intuitive-logical forms:

(i) The individual at the moment considered refuses to regard $a/h \prec b/k$.

(ii) Either it is asserted that a is true, or else it is asserted that b is true, or else that both a and b are true.

(iii) For each $n = 1, 2, \cdots$ it is asserted that a_n is true.

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[9] Two comments may be made in passing: Firstly, in view of Axiom I, $(\subset)$ may be replaced throughout by $(\prec)$; for $a \subset b$ which coincides with $a \sim b \subset 0$ is coextensive with $a \sim b/1 \prec 0/1$. Secondly, in the assertion $a/h \prec b/k$, on the left h is in a sense asserted, on the right on the other hand, k is regarded as asserted. Such temporary or localized assertions are not of the kind which form hypotheses or conclusions in our axioms or theorems; that is why we have distinguished them by the appellation of *presumptions*.

It is to be emphasized that this notation is introduced purely for convenience as an abbreviation: without it all the forthcoming results would subsist. We shall assume the intuitive-logical rules for dealing with these symbols, and use parenthesis in a self-explanatory manner.[10]

We now come to the point where a certain logical distinction must be explicitly noted: *A disjunction of asserted propositions is fundamentally different from the assertion of the corresponding disjunction of the contemplated propositions.* Thus if a and b are contemplated propositions, the asserted proposition $(a = 1) \vee (b = 1)$ is fundamentally different from the asserted proposition $(a \vee b) = 1$. Similarly, the two following assertions are absolutely different.

$$\sum_{n=1}^{\infty} (a_n = 1); \qquad (\sum_{n=1}^{\infty} a_n) = 1.$$

Thus when $(a = 1) \vee (b = 1)$ occurs in the hypothesis of a theorem, we can always split the theorem up into two theorems, the first with $a = 1$ in the hypothesis, the second with $b = 1$: What is common to the conclusion of both theorems is then taken as the conclusion of the original theorem with the hypothesis $(a = 1) \vee (b = 1)$. Similarly, if $\sum_n (a_n = 1)$ is given by hypothesis, to prove the conclusion amounts to proving that it follows from each hypothesis $a_n = 1$ $(n = 1, 2, \ldots)$. This is essentially different if the hypothesis is $(a \vee b) = 1$ in the first case, $(\sum_n a_n) = 1$ in the second. This point is so fundamental in the foundations of probability and has been so generally ignored that we shall consider in detail some of the paradoxes to which its disregard may lead.

Firstly there is the frequently heard objection to the possibility of any theory of probability, typified by the remark that proposition a is either true or false; in either case such an assertion as $a/1 \approx \sim a/1$ is untrue; hence it is an impossible assertion. To be sure, we always assert that a is either true or false in the sense that we always assert $(a \vee \sim a) = 1$; but in the above reasoning this is fallaciously confused with $(a = 1) \vee (\sim a = 1)$ and since the conclusion $\sim(a/1 \approx \sim a/1)$ is a consequence of $a = 1$ as well as of $\sim a = 1$, its general validity is regarded as established. Instances of like kind lead us to the view that the distinction upon which we are dwelling is indeed a *sine qua non* in the theory of probability.

Secondly, one might attempt the following proof of the Axiom of Alternative presumption (AAP §3), or better, of the following generalization:

HYPOTHESIS. (i) $a/hb_n \prec r/s$ $(n = 1, 2, \cdots)$.

(ii) $h \subset \sum_{n=1}^{\infty} b_n$, i.e., $(h \sum_{n=1}^{\infty} b_n) = h$.

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[10] Clearly we could carry the transcription into logical notation one step further and express all our axioms and theorems as implication statements between the $(\subset, \prec)$-assertions. Thus we would be in the presence of three distinct logical types of propositions: the contemplated propositions, the $(\subset, \prec)$-assertions, and the axioms and theorems of probability. But it is unnecessary and may be confusing to carry the symbolization so far in a work of this nature.

(iii) $b_i b_j = 0 (i \neq j)$.

CONCLUSION. $a/h \prec r/s$.

For (ii) can evidently be written

$$(\sum_{n=1}^{\infty} hb_n) = h,$$

which, if the fundamental distinction were not observed would be confused with

$$\text{(iii)} \quad \sum_{n=1}^{\infty} (hb_n = h), \text{ i.e., } \sum_{n=1}^{\infty} (h \subset b_n).$$

This latter can, as we have already explained, be split into the sequence of hypotheses $h \subset b_n$ $(n = 1, 2, \cdots)$. But in each case $hb_n = h$ and we have $a/h = a/hb_n \prec r/s$, and thus the conclusion, being true in each case, follows in general. The invalidity of this proof due to the confusion of (iii) with (ii) is what has forced us to list Axiom P as an unproved assumption. As a matter of fact the above infinite extension of Axiom P appears to be false, as the following example would show.

Consider two sequences of propositions $(a_1, a_2, \cdots)$ and $(a_1', a_2', \cdots)$ satisfying the conditions

$$(1) \qquad a = \sum_{n=1}^{\infty} a_n; \qquad a' = \sum_{n=1}^{\infty} a_n'; \qquad (a \vee a') = h \neq 0;$$

$$(2) \qquad \begin{aligned} a_i a_j &= a_i' a_j' = 0 \qquad (i \neq j); \\ a_i a_j' &= 0 \qquad (i, j = 1, 2, \cdots); \end{aligned}$$

$$(3) \qquad a_i/h \approx a_j/h \approx a_i'/h \approx a_j'/h.$$

No a_i or $a_j' = 0$ as this would lead by (3) and AAP Theorem 1 to this relation for all a_i, a_j', hence to $h = 0$ contrary to (1). Further, the evident relation $a_i \vee a_j' \vee a_k'/h \approx a_i \vee a_j' \vee a_k'/h$ in combination with (3) gives, by AAP Theorem 2 (i.e., Axiom D sharpened)

$$a_i/a_i \vee a_j' \vee a_k' \approx a_j'/a_i \vee a_j' \vee a_k' \approx a_k'/a_i \vee a_j' \vee a_k'$$

from which it appears (AAP Theorem 20) that $a_i/a_i \vee a_j' \vee a_k'$ is appraisable and of numerical probability 1/3. Hence, if r/s is appraisable and of numerical probability 5/12, we have $a_i/a_i \vee a_j' \vee a_k' \prec r/s$. On setting $b_n = a_n \vee a_{2n-1}' \vee a_{2n}'$ $(n = 1, 2, \cdots)$ and noting the relations $a/hb_n = ab_n/b_n = a_n/a_n a_{2n-1}' a_{2n}'$ we find that Hypothesis (i) of the theorem we are testing is verified; and (ii) is a consequence of (1), (iii) of (2). Hence the conclusion $a/h \prec r/s$, hence $p^*(a/h) \leqq 5/12$.

On replacing a etc. by a' etc. in the above reasoning, we obtain $p^*(a'/h) \leqq 5/12$. Hence by AAP Theorem 18 we have $p^*(h/h) = p^*(a \vee a'/h) \leqq p^*(a/h) + p^*(a'/h) \leqq 5/6$. And since $p^*(h/h) = 1$, we have a contradiction.

Returning to the sequence of trials of E, let a_n denote the contemplated experi-

mental proposition "the n'^{th} trial leads to success." We have now to cast into unambiguous logical form the definition of the statistical weight or frequency of success w. According to the First Assumption of Collectives: *If* $r(n)$ *is the number of the true propositions in the set* $(a_1, \cdots, a_n)$, *then* $\lim_{n\to\infty} r(n)/n = w$.

First of all, how is this definition of $r(n)$ to be understood? At the two extremes there are the following interpretations, denoted for brevity by $U(n, r)$ and $V(n, r)$:

$$U(n, r): \qquad \sum_{(s)} (a_{s_1} \cdots a_{s_r} \sim a_{s_{r+1}} \cdots \sim a_{s_n} = 1),$$

$$V(n, r): \qquad (\sum_{(s)} a_{s_1} \cdots a_{s_r} \sim a_{s_{r+1}} \cdots \sim a_{s_n}) = 1.$$

Here $(s) = (s_1, \cdots, s_n)$ denotes any permutation of $(1, \cdots, n)$ and the $\sum_{(s)}$ calls for the formation of all the terms for all possible (s) and their combination by disjunction $(\vee)$: that only $n!/r!(n-r)!$ of the terms are distinct is immaterial. But between these two extremes are a host of intermediate possibilities; thus in the case $n = 3$, $r = 2$ there are formulations such as

$$[(a_1a_2 \sim a_3 \vee a_1a_3 \sim a_2) = 1] \vee [a_2a_3 \sim a_1 = 1].$$

It will be realized at once that $V(n, r)$ is in a certain sense the weakest of all these expressions; it turns out to be the natural one for the theory of probability. But there is still a further modification in the direction of weakening which can be made, as we shall now see.

The explicit wording of the definition of w would now appear to be: "For every positive integer μ there exists a positive integer m such that assertion $V(n, r)$ holds for all $n = m, m + 1, m + 2, \cdots$, and all r between $nw - n/\mu$ and $nw + n/\mu$." This suggests the precise logical rendering:

$$\prod_{\mu=1}^{\infty} \sum_{m=1}^{\infty} \prod_{n=m}^{\infty} \sum_{nw-n/\mu}^{nw+n/\mu} (r)\, V(n, r).$$

But this is capable of further weakening; for

$$\sum_{nw-n/\mu}^{nw+n/\mu} (r)\, \{(\sum_{(s)} a_{s_1} \cdots a_{s_r} \sim a_{s_{r+1}} \cdots \sim a_{s_n}) = 1\}$$

is only one possible interpretation of the statement: "the number of true propositions in the set $(a_1, \cdots, a_n)$ is between $nw - n/\mu$ and $nw + n/\mu$." Another rendering is the assertion which we will call $W(w, \mu, n)$ suggested by modifying the preceding formula into

$$\{\sum_{nw-n/\mu}^{nw+n/\mu} (r) \sum_{(s)} a_{s_1} \cdots a_{s_r} \sim a_{s_{r+1}} \cdots \sim a_{s_n}\} = 1.$$

Some evident algebraic manipulations allow it to be put into a simpler form. Let $t = t(w, \mu, n)$ denote the least integer $\geqq nw - n/\mu$, and $f = f(w, \mu, n)$ the least integer $\geqq n(1 - w) - n/\mu$. Let

$$(p,\ q) = (p_1,\ \cdots,\ p_t,\ q_1,\ \cdots,\ q_f)$$

denote any set of $t + f$ distinct integers chosen from $(1, \cdots, n)$. Finally, employ $\sum(p, q)$ to denote the disjunction $(\vee)$ of all the terms formed by taking all possible choices of (p, q) $(w, \mu, n$ being fixed) in the summand. Then the definitive formulation of the weakest interpretation of the assertion "the number of true propositions in $(a_1, \cdots, a_n)$ is between $nw - n/\mu$ and $nw + n/\mu$" is as follows

$$W(w, \mu, n){:}\ (\sum_{(p,q)} a_{p_1} \cdots a_{p_t} \sim a_{q_1} \cdots \sim a_{q_f}) = 1.$$

(Note that only $n!/t!f!(n - t - f)!$ terms in the disjunction are distinct). In terms of this we have the assertion characterizing w:

$$W(w){:}\ \prod_{\mu=1}^{\infty} \sum_{m=1}^{\infty} \prod_{n=m}^{\infty} W(w, \mu, n).$$

We are now in a position to consider the following paradox: When $0 < w < 1$ there are infinitely many true propositions in $(a_1, a_2, \cdots)$ and infinitely many false; let the former be denoted in their order of occurrence by $(a_{\tau_1}, a_{\tau_2}, \cdots)$ and the latter by $(a_{\varphi_1}, a_{\varphi_2}, \cdots)$. Then any number θ $(0 \leqq \theta \leqq 1)$ being given, an explicit process is easily formulated which combines these two sequences into a single sequence—which is but a re-ordering of $(a_1, a_2, \cdots)$—for which the frequency of the a_{τ_i} is θ. How then can a frequency be taken as the numerical probability of success of E?

The whole question hinges on the meaning of a_{τ_1}. If the statement "a_{τ_1} is the first true proposition in $(a_1, a_2, \cdots)$" be taken to mean "true in Nature" i.e., as experimentally verifiable but of unknown verity, then the frequency of true propositions has nothing to do with the definition $W(w)$ of w. If it means "asserted as true," i.e., $= 1$, then $W(w)$ does not allow a_{τ_1} to be defined. For $W(w)$ allows us only to conclude that for some w, μ, n for which $t(w, \mu, n) > 0$, we have the assertion $W(w, \mu, n)$; in one sense this tells us that at least one a_i in the set $(a_1, \cdots, a_n)$ is true, but in precisely the same sense as the assertion $(a \vee \sim a) = 1$ tells us that at least one of $(a, \sim a)$ is true: it leaves the verbiage "the first proposition in $(a, \sim a)$ which is asserted as true" without meaning. In last analysis it is but the first paradox which we have considered in more complicated form.

If $W(w)$ were replaced by

$$\prod_{\mu=1}^{\infty} \sum_{m=1}^{\infty} \prod_{n=m}^{\infty} \sum_{(p,q)} (a_{p_1} \cdots a_{p_t} \sim a_{q_1} \cdots \sim a_{q_f} = 1)$$

the definition of $(a_{\tau_1}, a_{\tau_2}, \cdots)$ and $(a_{\varphi_1}, a_{\varphi_2}, \cdots)$ would be possible. Thus if $w = 1/2$, $\mu = 4$, and $n = 4\nu \geqq m$, we have

$$\sum_{(p,q)} (a_{p_1} \cdots a_{p_t} \sim a_{q_1} \cdots \sim a_{q_f} = 1).$$

Now t = least integer $\geqq nw - n/\mu = 2\nu - \nu = \nu$ so that $t = \nu$; similarly, $f = \nu$. Thus we have

$$(a_1 \cdots a_\nu \sim a_{\nu+1} \cdots \sim a_{2\nu} = 1) \vee \cdots \vee (a_{3\nu} \cdots a_{4\nu} \sim a_{2\nu} \cdots \sim a_{3\nu-1} = 1).$$

The disjoined terms are thought of as arranged lexicographically according to subscripts in the set of first factors such as $a_1 \cdots a_\nu$, which are themselves supposed to be written in increasing order of subscript. Thus we may speak of the first term asserted as true, i.e., such that $a_{p_1} \cdots a_{p_\nu} \sim a_{p_{\nu+1}} \cdots \sim a_{p_{2\nu}} = 1$, and then set $\tau_1 = p_1 : a_{\tau_1}$ is defined. Similarly for later terms.

Thus this paradox like the rest is resolved by maintaining the logical distinction which is the subject of this section.

Before closing we shall consider an extension. Let h be the common presumption made in envisaging the outcomes of E: We are considering the sequence of eventualities a_n/h. We wish to formulate an assertion $W_h(w)$ which is the extension of $W(w)$ and characterizes w regarded as the frequency of successes of E on the presumption that h is true. Such an assertion is evidently furnished by replacing $(\cdots) = 1$ in the above by the assertion $h \subset (\cdots)$ (or equivalently $h \cdot (\cdots) = h$). Everything else being as above, we set

$$W_h(w, \mu, n)\colon h \subset \sum_{(p,\,q)} a_{p_1} \cdots a_{p_t} \sim a_{q_1} \cdots \sim a_{q_f}$$

and correspondingly,

$$W_h(w)\colon \prod_{\mu=1}^{\infty} \sum_{m=1}^{\infty} \prod_{n=m}^{\infty} W_h(w, \mu, n).$$

Finally, if $\sum_{0 \leqq w \leqq 1}$ is used for the disjunction of all the terms formed by letting w range from 0 to 1, the assertion: "there is a frequency of successes on the presumption h" shall be formulated as

$$W_h\colon \sum_{0 \leqq w \leqq 1} W_h(w).$$

3. Symmetry and Independence

DEFINITION 1. *Any finite or countable set of propositions $(a_1, a_2, \cdots)$ shall be said to be a symmetric set with respect to the proposition $h(\neq 0)$ when for every possible positive integral t*

$$(1) \qquad a_{i_1} \cdots a_{i_t}/h \approx a_{j_1} \cdots a_{j_t}/h$$

holds for every choice of the t distinct integers $(i_1, \cdots, i_t)$ and the t distinct integers $(j_1, \cdots, j_t)$.

Here $h \neq 0$ denotes the assertion $\sim (h = 0)$.

Let $\varphi(x_1, \cdots, x_m)$ denote a finite combination of the symbols $x_1, \cdots, x_m$ with the aid of the logical constants $(\sim \cdot \vee)$, the convention being that the formal laws of Boolean algebra hold for these symbols; such a $\varphi(x_1, \cdots, x_m)$ may be called a Boolean polynomial in these m (propositional) variables. If by

application of the formal Boolean identities alone, i.e., without assuming any individual properties of $x_1, \cdots, x_m$, it may be reduced to 0, 1, or some other given proposition not involving the variables, we will say that it is identically equal ($\equiv$) to this proposition.

Theorem 1. *If $(a_1, a_2, \cdots)$ is a symmetric set with respect to h, then for any $\varphi(x_1, \cdots, x_m)$ (m not exceeding the number of propositions in the set) the relation*

$$\varphi(a_{i_1}, \cdots, a_{i_m})/h \approx \varphi(a_{j_1}, \cdots, a_{j_m})/h \tag{2}$$

holds for every possible pair $(i_1, \cdots, i_m)$ and $(j_1, \cdots, j_m)$ of m distinct positive integers each.

Elementary algebraic manipulations show that any $\varphi(x_1, \cdots, x_m)$ can be expressed as the disjunction of disjoint monomials $x_{k_1} \cdots x_{k_t} \sim x_{k_{t+1}} \cdots \sim x_{k_{t+s}}$; thus when (2) has been established for monomial $\varphi(x_1, \cdots, x_m)$ its general validity becomes an immediate consequence of the theorem of total probability (AAP Theorem 6, sharpened). The relation

$$a_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots \sim a_{i_{t+s}}/h \approx a_{j_1} \cdots a_{j_t} \sim a_{j_{t+1}} \cdots \sim a_{j_{t+s}}/h$$

is proved by induction for all s. Assume it true for all values of the first subscript t and all values of the second between 0 and s inclusive and apply AAP Theorem 7 (sharpened) to

$$a_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots \sim a_{i_{t+s}} a_{i_{t+s+1}}/h \approx a_{j_1} \cdots a_{j_t} \sim a_{j_{t+1}} \cdots \sim a_{j_{t+s}} a_{j_{t+s+1}}/h,$$

$$(a_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots \sim a_{i_{t+s}}, \quad a_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots \sim a_{i_{t+s}} a_{i_{t+s+1}}, h$$

respectively replacing a_1, b_1, h_1, etc.). The required relation is an immediate consequence.

Corollary. *If $(a_1, a_2, \cdots)$ is symmetric with respect to h, then any subset of $(a_1, a_2, \cdots)$ and any subset of $(\sim a_1, \sim a_2, \cdots)$ is likewise.*

Definition 2. *Any finite or countable set of propositions $(a_1, a_2, \cdots)$ shall be said to be an independent set with respect to the proposition $h(\neq 0)$ if firstly $ha_i \neq 0$ or h, and secondly*

$$a_i/ha_{i_1} \cdots a_{i_t} \approx a_i/h \tag{3}$$

for every possible t and every choice of $t + 1$ distinct integers $(i, i_1, \cdots, i_t)$.

We must prove that (3) has meaning in every case, i.e., that $ha_{i_1} \cdots a_{i_t} \neq 0$. This is assumed in the hypothesis when $t = 1$. Proceeding inductively, let it be assumed for the value t; we then will show that $ha_{i_1} \cdots a_{i_{t+1}} \neq 0$. If $ha_{i_1} \cdots a_{i_{t+1}} = 0$ the following relations would occur:

$$a_{i_{t+1}}/h \approx a_{i_{t+1}}/ha_{i_1} \cdots a_{i_t} = ha_{i_1} \cdots a_{i_{t+1}}/ha_{i_1} \cdots a_{i_t} = 0/ha_{i_1} \cdots a_{i_t}$$

and hence $ha_{i_{t+1}} = 0$ (AAP Theorem 1) contrary to hypothesis.

A precisely similar process shows that $ha_{i_1} \cdots a_{i_t} \neq h$, and furthermore that if $(i_1, \cdots, i_{t+s})$ are any $t + s$ distinct positive integers

$$ha_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots \sim a_{i_{t+s}} \neq 0 \text{ or } h.$$

From this relation Theorem 2 follows immediately:

THEOREM 2. *If* $(a_1, a_2, \cdots)$ *is an independent set with respect to* h *and if* $\varphi(x_1, \cdots, x_m)$ *is any Boolean polynomial not identically* 0 *or* h, *then for every* m *distinct* $(i_1, \cdots, i_m)$, $h\varphi(a_{i_1}, \cdots, a_{i_m}) \neq 0$ *or* h.

THEOREM 3. *If* $(a_1, a_2, \cdots)$ *is an independent set with respect to* h *and if* $\varphi(x_1, \cdots, x_m)$ $(\not\equiv 0, h)$ *is as above, then for any* $m + 1$ *distinct* $(i, i_1, \cdots, i_m)$

$$a_i/h\varphi(a_{i_1}, \cdots, a_{i_m}) \approx a_i/h. \tag{4}$$

As in the proof of Theorem 1, $\varphi(x_1, \cdots, x_m)$ is expressed as the disjunction of disjoint monomials $x_{k_1} \cdots x_{k_t} \sim x_{k_{t+1}} \cdots \sim x_{k_{t+s}}$. Once the relation

$$a/ha_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots \sim a_{i_{t+s}} \approx a_i/h \tag{5}$$

has been established the truth of (4) will follow by an immediate inductive application of Axiom P.[11]

We assume the truth of (5) for all values of the first index t and all values of the second between 0 and s inclusive, and seek to prove it when the second index is replaced by $s + 1$. On account of the assumption we have

$$a_i/ha_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots \sim a_{i_{t+s}} a_{i_{t+s+1}} \approx a_i/h$$
$$\approx a_i/ha_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots \sim a_{i_{t+s}}.$$

The desired relation follows on applying AAP Theorem 10 (sharpened) with a_i, $a_{i_{t+s+1}}$, $ha_{i_1} \cdots a_{i_t} \sim a_{i_{t+1}} \cdots a_{i_{t+s}}$ replacing the a, k, h, respectively, of that theorem.

COROLLARY. *If* $(a_1, a_2, \cdots)$ *is an independent set with respect to* h, *then any subset of* $(a_1, a_2, \cdots)$ *and any subset of* $(\sim a_1, \sim a_2, \cdots)$ *is likewise.*

The notion of symmetry corresponds with that of events all equally probable and such that any combination of results of one set is as probable of occurrence as the same combination of another set. The notion of independence on the other hand corresponds with the idea that knowledge of the occurrence of some of the events does not change the probability ascribed to that of the others. A set can be symmetric without being independent, for symmetry does not exclude $h\varphi(a_1, \cdots, a_m) = 0, h$: sets in which $a_1 = a_2 = \cdots$ or in which $a_1 \vee \cdots \vee a_m = 0$ may be symmetric but can not be independent. On the other hand sets may be independent without being symmetric; for example a_1, a_2, etc. may denote success of respective trials of unconnected events but events of different kind and thus in general, different probability. Hence the significance of the following

[11] The theorem employed here is the following:

If $hc_ic_j = 0$ *for all* $i \neq j$ *and if* $a/hc_i \prec r/s$ $(i = 1, \cdots, n)$, *then* $a/hc \prec r/s$ *where* $c = c_1 \vee \cdots \vee c_n$.

This is proved inductively from the case $n = 2$, in which it is a self-evident consequence of Axiom $P(c_1$ and $h(c_1 \vee c_2)$ replacing b and $h)$. The sharpening and strengthenings are immediate. This theorem belongs properly in AAP §4, just before Theorem 13.

THEOREM 4. *If* $(a_1, a_2, \cdots)$ *is an independent set with respect to h and if* $a_i/h \approx a_j/h$ *for all i and j, then* $(a_1, a_2, \cdots)$ *is a symmetric set with respect to h.*

For we have by hypothesis

$$a_{i_1}/ha_{i_2} \approx a_{i_1}/h \approx a_{i_2}/h \approx a_{j_2}/h \approx a_{j_1}/h \approx a_{j_1}/haj_2 .$$

On applying Axiom C_1 (sharpened by AAP Theorem 2) with $h, a_{i_2}, a_{i_1}, h, a_{j_2}, a_{j_1}$ in lieu of $h_1, a_1, b_1, h_2, a_2, b_2$, the result $a_{i_1}a_{i_2}/h \approx a_{j_1}a_{j_2}/h$ is obtained; the proof of the general relation (1) proceeds by induction.

In the classical theory the only relations of probability are those concerned with numerical probability. The notions of symmetry and independence are *less* restictive insofar as equiprobability $(a/h \approx b/k)$ implies equal numerical probability $(p(a/h) = p(b/k))$ and not conversely; but they are *more* restrictive in that the appraisability of every eventuality is assumed. We give the following definitions and theorems (in essence familiar) mainly for completeness. The proofs are supplied immediately by means of the laws of total and compound probability (cf. AAP, §5, Theorems 23 to 28).

Let $(a_1, a_2, \cdots)$ be any finite or countable set of propositions, let $h \neq 0$ be a further proposition, and assume that all the eventualities with respect to h (i.e., remainder classes with respect to the principle ideal $(\sim h)$ in the Boolean ring determined by $(h_1, a_1, a_2, \cdots)$) are appraisable.

DEFINITION 3. *The set* $(a_1, a_2, \cdots)$ *shall be called numerically symmetric with respect to h when for all indices as in Definition* 1,

$$p(a_{i_1} \cdots a_{i_t}/h) = p(a_{j_1} \cdots a_{j_t}/h). \tag{5}$$

THEOREM 5. *If* $(a_1, a_2, \cdots)$ *is a numerically symmetric set with respect to h, then for any Boolean polynomial* $\varphi(x_1, \cdots, x_m)$

$$p(\varphi(a_{i_1}, \cdots, a_{i_m})/h) = p(\varphi(a_{j_1}, \cdots, a_{j_m})/h).$$

COROLLARY. *The numerical symmetry with respect to h of* $(a_1, a_2, \cdots)$ *implies that of any sub-set of* $(a_1, a_2, \cdots)$ *and any sub-set of* $(\sim a_1, \sim a_2, \cdots)$.

DEFINITION 4. *The set* $(a_1, a_2, \cdots)$ *shall be said to be numerically independent with respect to h if firstly* $0 < p(a_i/h) < 1$ *for all i, and secondly (the indices being as in Definition 2),*

$$p(a_i/ha_{i_1} \cdots a_{i_t}) = p(a_i/h). \tag{6}$$

THEOREM 6. *If* $(a_1, a_2, \cdots)$ *is numerically independent with respect to h and* $\varphi(x_1, \cdots, x_m)$ *any Boolean polynomial* $(\not\equiv 0, h)$ *then for every m distinct* $(i_1, \cdots, i_m)$

$$0 < p(\varphi(a_{i_1}, \cdots, a_{i_m})/h) < 1.$$

THEOREM 7. *If* $(a_1, a_2, \cdots)$ *and* $\varphi(x_1, \cdots, x_m)$ *are as in Theorem* 6, *then for any* $m + 1$ *distinct* $(i, i_1, \cdots, i_m)$

$$p(a_i/h\varphi(a_{i_1}, \cdots, a_{i_m})) = p(a_i/h).$$

COROLLARY 1. *Under these conditions* $p(a_{i_1} \cdots a_{i_m}/h) = p(a_{i_1}/h)p(a_{i_2}/h) \cdots p(a_{i_m}/h)$.

COROLLARY 2. *The numerical independence with respect to h of* $(a_1, a_2, \cdots)$ *implies that of any sub-set of* $(a_1, a_2, \cdots)$ *and of any sub-set of* $(\sim a_1, \sim a_2, \cdots)$.

COROLLARY 3. *If* $(a_1, a_2, \cdots)$ *is a numerically independent set with respect to h and if for all i and j,* $p(a_i/h) = p(a_j/h)$, *then it is numerically symmetric.*

4. Theorems on Sequences

THEOREM 8. *If* $(a_1, \cdots, a_n)$ *is symmetric with respect to h and if precisely* $r(0 \leqq r \leqq n)$ *of its elements are implied by h in the sense that* (§2)

$$V_h(n, r)\colon h \subset \sum_{(s)} a_{s_1} \cdots a_{s_r} \sim a_{s_{r+1}} \cdots \sim a_{s_n},$$

then a_i/h *is appraisable and* $p(a_i/h) = r/n$ $(i = 1, \cdots, n)$.

There is no loss of generality in assuming that $a_i \subset h$, i.e., $ha_i = a_i$ $(i = 1, \cdots, n)$; for this property can always be secured by replacing $(a_1, \cdots, a_n)$ by $(ha_1, \cdots, ha_n)$ which evidently also satisfies the hypothesis; since $ha_i/h = a_i/h$ the conclusions are the same. Proceeding on this assumption and excluding the trivial case of our theorem in which $r = 0$, we have on multiplying hypothesis $V_h(n, r)$ through by h

$$h = \sum_{(s)} a_{s_1} \cdots a_{s_r} \sim a_{s_{r+1}} \cdots \sim a_{s_n}.$$

The summation $\sum_{(s)}$ is for all permutations $(s) : (s_1, \cdots, s_n)$ of $(1, \cdots, n)$, as above.

There are evidently $\nu = n!/r!(n - r)!$ distinct terms in this equation (i.e., terms not reducible to one another by the identities of Boolean algebra); let them be denoted in some conventional order by $(u_1, \cdots, u_\nu)$. Of these terms there will be clearly $\tau = (n - 1)!/(r - 1)!\,(n - r)!$ distinct ones involving a preassigned a_i as factor; let them be $(u_{i_1}, \cdots, u_{i_\tau})$. We have moreover $u_k u_l = 0$ for all $k \neq l$, and of course $u_1 \vee \cdots \vee u_\nu = h \neq 0$. Finally, as a result of the symmetry (cf. Theorem 1) $u_k/h \approx u_l/h$ for all k and l. Thus $(u_1, \cdots, u_\nu)$ forms a ν-scale (AAP, §5, Definition 1) and if we set $b = u_{i_1} \vee \cdots \vee u_{i_\tau}$ we have by AAP Theorem 20 that b/h is appraisable and $p(b/h) = \tau/\nu = r/n$. Thus our theorem will be proved once it is established that $b/h = a_i/h$, i.e., that $hb = ha_i$. This is shown on multiplying the equation

$$u_1 \vee \cdots \vee u_\nu = b \vee u_{j_1} \vee \cdots \vee u_{j_{\nu-\tau}} = h$$

through by a_i : clearly if u_{j_1} does not have a_i as a factor, it does have $\sim a_i$, so that $a_i u_{j_1} = 0$, etc.

COROLLARY. *If* $(a_1, \cdots, a_n)$, *h and* $(a'_1, \cdots, a'_n)$, *h' each satisfy the hypothesis of Theorem 8 for the same value of r, then* $a_i/h \approx a'_i/h'$.

THEOREM 9. *If* $(a_1, \cdots, a_n)$ *is symmetric with respect to h and if at least r* $(0 \leqq r \leqq 1)$ *of its elements are implied by h in the sense that*

$$X_h(n, r)\colon h \subset \sum_{(s_1, \cdots, s_r)} a_{s_1} \cdots a_{s_r}$$

(where the summation takes in all possible sets of r distinct integers in $(1, \cdots, n)$*), then for each value of i,* $p_*(a_i/h) \geqq r/n$.

Again we assume $a_i \subset h$ $(i = 1, \cdots, n)$ and disregard the trivial case $r = 0$. Multiplication by h puts $X_h(n, r)$ into the form $h = \sum a_{s_1} \cdots a_{s_r}$. We set

$$c_\rho = \sum_{(s)} a_{s_1} \cdots a_{s_\rho} \sim a_{s_{\rho+1}} \cdots \sim a_{s_n}; \tag{1}$$

here as in $V_h(n, r)$ the summation is for all permutations (s): $(s_1, \cdots, s_n)$ of $(1, \cdots, n)$. Simple algebraic manipulations establish

$$h = \sum_{\rho=r}^{n} c_\rho; \qquad c_\rho c_\sigma = 0 \ (\rho \neq \sigma); \qquad hc_\rho = c_\rho. \tag{2}$$

Let ρ be a value for which $c_\rho \neq 0$: since $h \neq 0$, such values will occur. We will show that $(a_1, \cdots, a_n)$ is symmetric with respect to c_ρ. In order to prove

$$a_{i_1} \cdots a_{i_t}/c_\rho \approx a_{j_1} \cdots a_{j_t}/c_\rho \tag{3}$$

we first introduce the Boolean polynomial

$$\varphi(x_1, \cdots, x_n) \equiv x_1 \cdots x_t \sum_{(s)} x_{s_1} \cdots x_{s_\rho} \sim x_{s_{\rho+1}} \cdots \sim x_{s_n}$$

and apply Theorem 1, whereupon the equation

$$\varphi(x_{i_1}, \cdots, x_{i_t}, x_{i_{t+1}}, \cdots, x_{i_n})/h \approx \varphi(x_{j_1}, \cdots, x_{j_t}, x_{j_{t+1}}, \cdots, x_{j_n})/h$$

is obtained (all subscripts in each set being distinct). But this may be written as

$$c_\rho a_{i_1} \cdots a_{i_t}/h \approx c_\rho a_{j_1} \cdots a_{j_t}/h. \tag{4}$$

Now if either $c_\rho a_{i_1} \cdots a_{i_t} = 0$ or $c_\rho a_{j_1} \cdots a_{j_t} = 0$, (4) shows that both these equations hold, and (3) becomes evident. In all other cases we apply Axiom D (sharpened) to (4) together with $c_\rho/h \approx c_\rho/h$ (with $a_1 = a_2 = c_\rho$, $h_1 = h_2 = h$, $b_1 = a_{i_1} \cdots a_{i_t}$, $b_2 = a_{j_1} \cdots a_{j_t}$) whereupon (3) is established.

This symmetry and the relation (1) show that $(a_1, \cdots, a_n)$ satisfies the hypothesis of Theorem 8 with (ρ, c_ρ) replacing (r, h); it follows that a_i/c_ρ is appraisable and $p(a_i/c_\rho) = \rho/n$ $(i = 1, \cdots, n)$.

Let σ be any given positive integer, and $(u_1, \cdots, u_{n\sigma})$ an $n\sigma$-scale. Set $v = u_1 \vee \cdots \vee u_{r\sigma-1}$, $u = u_1 \vee \cdots \vee u_{n\sigma}$; then $p(v/u) = (r\sigma - 1)/n\sigma = r/n - 1/n\sigma$ (AAP Theorem 20). Hence for each ρ for which $c_\rho \neq 0$ we have $a_i/c_\rho \succ v/u$ (AAP Theorem 16).

Returning to (2), we may write $h = \sum' c_\rho$ where the summation $\sum'$ includes only those terms of $(c_r, \cdots, c_n)$ which $\neq 0$. By applying Axiom P[11] to $a_i/c_\rho \succ v/u$ we derive the result that $a_i/\sum' c_\rho = a_i/h \succ v/u$. Hence by definition of $p_*(a_i/h)$ we have for all values of the positive integer σ that $p_*(a_i/h) \geqq r/n - 1/n\sigma$, which leads at once to the conclusion of our theorem.

THEOREM 10. *If* $(a_1, \cdots, a_n)$ *is symmetric with respect to h and if h implies the truth of at least t and falsehood of at least f of its elements in the sense that*

$$X_h(n, t, f): h \subset \sum_{(p, q)} a_{p_1} \cdots a_{p_t} \sim a_{q_1} \cdots \sim a_{q_f}$$

(where the summation takes in all possible sets of $t + f$ *distinct integers from* $(1, \cdots, n)$*), then*

$$\frac{t}{n} \leqq p_*(a_i/h) \leqq p^*(a_i/h) \leqq 1 - \frac{f}{n}. \tag{5}$$

The first inequality in (5) results from the application of Theorem 9 ($r = t$) inasmuch as it is clear that

$$h \subset \sum_{(p, q)} a_{p_1} \cdots a_{p_t} \sim a_{q_1} \cdots \sim a_{q_f} \subset \sum_{(s)} a_{s_1} \cdots a_{s_t}.$$

To obtain the last inequality in (5) we have but to replace $(a_1, \cdots, a_n)$ by $(\sim a_1, \cdots, \sim a_n)$ (cf. Theorem 1, Corollary) and apply AAP Theorem 17 to the resulting inequality

$$f/n \leqq p_*(\sim a_i/h) = 1 - p^*(a_i/h).$$

THEOREM 11. *Let the infinite sequence* $(a_1, a_2, \cdots)$ *be a symmetric set with respect to h, and let the limiting frequency (statistical weight) of its elements implied by h be w in the sense of the assertion*

$$W_h(w): \prod_{\mu=1}^{\infty} \sum_{m=1}^{\infty} \prod_{n=m}^{\infty} W_h(w, \mu, n),$$

in which $W_h(w, \mu, n)$ *coincides with the assertion* $X_h(n, t, f)$ *in Theorem 10 in which* $t = t(w, \mu, n)$ *is the least integer* $\geqq nw - n/\mu$ *and* $f = f(w, \mu, n)$ *is the least integer* $\geqq n(1 - w) - n/\mu$.

Then a_i/h *is appraisable and* $p(a_i/h) = w$ $(i = 1, 2, \cdots)$.

Let us choose a value of μ and hold it fast. Then in virtue of the hypothesis $W_\mu(w)$ there exists an integer m for which $\prod_{n=m}^{\infty} W_h(w, \mu, n)$ is valid. For definiteness we select the least m for which this is true (we may as well represent it by m); then we have for this m and, e.g., for $n = m$ the assertion $W_h(w, \mu, n)$. Hence by Theorem 10 we conclude that

$$\frac{t(w, \mu, n)}{n} \leqq p_*(a_i/h) \leqq p^*(a_i/h) \leqq 1 - \frac{f(w, \mu, n)}{n}.$$

This yields on referring to the definitions of $t(w, \mu, n)$ and $f(w, \mu, n)$

$$w - \frac{1}{\mu} \leqq p_*(a_i/h) \leqq p^*(a_i/h) \leqq w + \frac{1}{\mu}.$$

This equation, being true for every $\mu = 1, 2, \cdots$, leads at once to the conclusion of our theorem.

COROLLARY. *Let* $(a_1, a_2, \cdots)$ *be symmetric with respect to h and let the limiting frequency of its elements implied by h exist and lie between* w_1 *and* w_2 *in the sense of the assertion* $\sum_{w_1 \leqq w \leqq w_2} W_h(w)$. *Then* a_i/h *is appraisable and* $w_1 \leqq p(a_i/h) \leqq w_2$ $(i = 1, 2, \cdots)$.

THEOREM 12. *Let $(a_1, a_2, \cdots)$ be symmetric with respect to h and $(b_1, b_2, \ldots)$ be symmetric with respect to k. Let the number of a's implied by h eventually become and remain greater than the number of b's implied by k in the sense of the assertion*

$$\Omega: \qquad \sum_{m=0}^{\infty} \prod_{n=m}^{\infty} \sum_{r=0}^{n-1} \{(h \subset \sum a_{s_1} \cdots a_{s_{r+1}})(k \subset \sum \sim b_{t_1} \cdots \sim b_{t_{n-r}})\}.$$

Then for every i, j: $a_i/h \succ b_j/k$.

As usual $(s_1, \cdots, s_{r+1})$ is a set of $r + 1$ distinct integers chosen from $(1, \cdots, n)$, all possible sets of this kind being taken in the $\sum$; similarly for $(t_1, \cdots, t_{n-r})$ and the corresponding $\sum$.

Let m denote an integer for which $\prod_{n=m}^{\infty} \cdots$ is true by assertion (for definiteness, the *least* m). Let n denote the greatest of (m, i, j). With this determination of n we have the assertion $\sum_{r=0}^{n-1} \{\ \}$. Let r denote a value for which assertion $\{\ \}$ is made. Then we have simultaneously

$$h \subset \sum a_{s_1} \cdots a_{s_{r+1}}, \qquad k \subset \sum \sim b_{t_1} \cdots \sim b_{t_{n-r}}.$$

We are now in a position to apply Theorem 9 to each of the sets $(a_1, \cdots, a_n)$ and $(\sim b_1, \cdots, \sim b_n)$, the latter of which is symmetric by the corollary to Theorem 1, and so obtain

$$p_*(a_i/h) \geqq \frac{r+1}{n}, \qquad p_*(\sim b_j/k) \geqq \frac{n-r}{n}.$$

But $p_*(\sim b_j/k) = 1 - p^*(b_j/k)$, and hence $p_*(a_i/h) > p^*(b_j/k)$, from which the conclusion of our theorem follows (AAP Theorems 17 and 16).

In concluding this section we may consider the question of replacing in the hypotheses of these theorems the requirement of symmetry by that of numerical symmetry. At first sight such a paraphrase would have the advantage of weakening the condition of symmetry; but this advantage is in our opinion far offset by the disadvantage that appraisability has to be assumed in the hypothesis—and it is indeed one of the chief functions of the theorems of this section that they afford the property of appraisability in their conclusions. Furthermore, symmetry rather than numerical symmetry appears to be the natural assumption to make in the applications. Nevertheless the quantitative conclusions subsist when numerical symmetry is assumed instead of symmetry. We confine ourselves to a statement of the following examples. The proofs, which proceed along rather similar lines to those of the earlier theorems and with the use of the classical properties of numerical probability, are omitted.

THEOREM 13. *If $(a_1, \cdots, a_n)$ is numerically symmetric with respect to h and if assertion $X_h(n, t, f)$ is made, then $t/n \leqq p(a_i/h) \leqq 1 - f/n$.*

THEOREM 14. *If $(a_1, a_2, \cdots)$ is numerically symmetric with respect to h and if assertion $W_h(w)$ is made, then $p(a_i/h) = w$.*

THEOREM 15. *If $(a_1, a_2, \cdots)$ is numerically symmetric with respect to h and*

$(b_1, b_2, \cdots)$ *is numerically symmetric with respect to* k *and if assertion* Ω *is made, then* $p(a_i/h) \geqq p(b_j/k)$.

In the theorems of this section the symmetry rather than the independence of the sets of propositions $(a_1, a_2, \cdots)$ appears to be the important notion. That such an appearance is deceptive when regard is had to the applications (the trials of event E) is realized when the possibility is considered that the symmetry of $(a_1, a_2, \cdots)$ with respect to h does not guarantee that of $(a_{n+1}, a_{n+2}, \cdots)$ with respect to $ha_1 \cdots a_n$: i.e., the results of the first n trials of E being known, we may not be in a position to apply our theorems to the others. Thus it would appear that the hypothesis of Theorem 4 affording both symmetry and independence is prescribed in such applications.

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OPERATOR-THEORETICAL TREATMENT OF MARKOFF'S PROCESS AND MEAN ERGODIC THEOREM

BY KÔSAKU YOSIDA AND SHIZUO KAKUTANI

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CHAPTER I. INTRODUCTION

It is the purpose of the present paper to give a consistent treatment of the problems of Markoff's process and mean ergodic theorem from the standpoint of the theory of iteration of bounded linear operations in Banach spaces.

Let Ω be an abstract space where a measure of Lebesgue type is defined, and consider a simple Markoff's process, by which a moving point $t \in \Omega$ is transferred stochastically inside Ω. If we denote by $P(t, E)$ the transition probability that a moving point $t \in \Omega$ is transferred into a Borel set E of Ω after the elapse of a unit-time, then we have always $P(t, E) \geqq 0$ and $P(t, \Omega) = 1$. We shall further assume that $P(t, E)$ is completely additive as a set function of Borel sets E if t is fixed, and that $P(t, E)$ is Borel measurable in t if E is fixed. Under these assumptions, the probability $P^{(n)}(t, E)$ that a moving point $t \in \Omega$ is transferred into a Borel set E of Ω after the elapse of n unit-times is given recurrently by

$$P^{(n)}(t, E) = \int_\Omega P^{(n-1)}(t, ds)P(s, E), \qquad n = 2, 3, \cdots; \qquad P^{(1)}(t, E) = P(t, E),$$

where the integration is of Radon-Stieltjes type.

The problem of Markoff consists in the investigation of the behavior of the sequence of the iterations $P^{(n)}(t, E)$ and their arithmetic means $Q^{(n)}(t, E) = \frac{1}{n}\sum_{m=1}^{n} P^{(m)}(t, E)$ for large n.

In the special case, when Ω is composed of only a finite number ($= N$) of points $\left(\text{with equal measure } \frac{1}{N}\right)$, our Markoff's process is reduced to the classical one, and our problem is nothing but the investigation of the behavior of the iterations P^n of a matrix $P = (p_{ij})$, $i, j = 1, 2, \cdots, N$, such that $p_{ij} \geqq 0$, $i, j = 1, 2, \cdots, N$, and $\sum_{j=1}^{N} p_{ij} = 1$, $i = 1, 2, \cdots, N$. This case was first treated by A. Markoff in his classical papers, and in the last ten years many contributions to this problem were given by J. Hadamard, M. Fréchet, R. von Mises, A. Kolmogoroff, W. Doeblin, J. L. Doob and others. These results are all collected in a recent monograph of M. Fréchet [3], which is so complete that we find almost no need of giving further discussions on this case. We shall only remark that this case may be treated exactly in the same manner as in the general case.

In the general case, when Ω is a continuum,[1] this problem was discussed by

[1] We shall confine ourselves in this paper only to the case when Ω is the closed interval $0 \leqq t \leqq 1$. This is not an essential restriction.

B. Hostinsky, M. Fréchet, J. L. Doob, W. Doeblin and N. Kryloff-N. Bogolioùboff. B. Hostinsky [1] treated the case when $P(t, E)$ is given by a continuous density $p(t, s)$: $P(t, E) = \int_E p(t, s)\, ds$, where $p(t, s)$ is a continuous function of two variables t and s in $\Omega \times \Omega$; M. Fréchet [1], [2] considered the case when $p(t, s)$ is bounded and measurable instead of being continuous; J. L. Doob [1] weakened this condition by requiring only that $p(t, s)$ is uniformly (in t) integrable in s, that is, that there exists for any $\epsilon > 0$ an $\eta > 0$ such that $\text{mes}(E) < \eta$ implies $\int_E p(t, s)\, ds < \epsilon$ for any $t \in \Omega$; and lastly, W. Doeblin [1] discussed the case when the existence of the density $p(t, s)$ is not necessarily assumed. He has only assumed that

$$\text{(D)} \quad \begin{cases} \textit{there exist an integer } d \textit{ and positive constants } b, \eta \textit{ such that } \text{mes}(E) < \eta \\ \textit{implies } P^{(d)}(t, E) \leqq 1 - b \textit{ for any } t \in \Omega. \end{cases}$$

It is clear that the condition (D) of W. Doeblin is more general than that of J. L. Doob, and under this condition, W. Doeblin has obtained, by a direct (point-set-theoretical) method, a considerably precise result.

In the present paper, we shall discuss these problems by another method, which is due to N. Kryloff-N. Bogolioùboff [1], [2], and which may be called an operator-theoretical method. Indeed, we shall treat these problems by considering $P^{(n)}(t, E)$ as a kernel of an integral operator in some special Banach spaces. This may be done in two ways: firstly,

$$x \to T(x) = y: \quad y(E) = \int_\Omega x(dt)P(t, E)$$

is a bounded positive[2] linear operation which maps the Banach space (**M**)[3] into itself; and secondly,

$$x \to \overline{T}(x) = y: \quad y(t) = \int_\Omega P(t, ds)y(s)$$

is a bounded positive linear operation which maps the Banach space (M^*)[4]

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[2] A bounded linear operation T, which maps a semi-ordered Banach space into itself, is called to be positive if $x \geqq 0$ implies $T(x) \geqq 0$.

[3] (**M**) is the space of all the (real- or complex-valued) completely additive set functions $x(E)$ defined for all Borel set E of Ω. For any $x(E) \in$ (**M**), its norm is defined by: $\| x \| =$ total variation of $| x(E) |$ on Ω. In (**M**), a real-valued completely additive set function $x(E)$ is called to be positive (and is denoted by $x \geqq 0$), if $x(E) \geqq 0$ for any Borel set $E \subset \Omega$.

[4] (M^*) is the space of all the (real- or complex-valued) bounded Borel measurable functions $x(t)$ defined on Ω. For any $x(t) \in (M^*)$, its norm is defined by: $\| x \| = \text{l.u.b.}_{t \in \Omega} | x(t) |$. In (M^*), two functions, which differ from each other at least at one point, are considered to be different; it is to noted that the Banach space (M^*) is essentially different from (M), where the norm is defined by: $\| x \| = \text{ess. max.}_{t \in \Omega} | x(t) |$. In (M^*), a real-valued bounded Borel measurable function $x(t)$ is called to be positive if $x(t) \geqq 0$ for any $t \in \Omega$.

into itself. If, in particular, $P(t, E)$ has the density $p(t, s)$, then these operations become

$$x \to T(x) = y: \quad y(E) = \int_E \left(\int_\Omega x(dt) p(t, s) \right) ds,$$

$$x \to \overline{T}(x) = y: \quad y(t) = \int_\Omega p(t, s) x(s)\, ds;$$

consequently, in the former case, T may also be considered as a bounded positive linear operation which maps the Banach space (L)[5] into itself:

$$x \to T(x) = y: \quad y(s) = \int_\Omega x(t)\, p(t, s)\, dt.$$

In all these cases, T and $\overline{T}$ are both of norm 1, and it will be easily seen that the iterates T^n and $\overline{T}^n$ of T and $\overline{T}$ respectively, which are defined by the iterated kernel $P^{(n)}(t, E)$:

$$x \to T^n(x) = y: \quad y(E) = \int_\Omega x(dt)\, P^{(n)}(t, E),$$

$$x \to \overline{T}^n(x) = y: \quad y(t) = \int_\Omega P^{(n)}(t, ds)\, x(s),$$

have also the same properties. Thus our problem is transformed into the investigation of the behavior of the iterates T^n of a bounded positive linear operation T, which maps the Banach space $(\mathbf{M})$, (L) or (M^*) into itself.

N. Kryloff-N. Bogolioùboff [1], [2] introduced the condition that

$$\text{(K)} \qquad \begin{cases} \textit{there exist an integer } m \textit{ and a strongly completely continuous}^6 \textit{ linear} \\ \textit{operation } V, \textit{ which maps } (\mathbf{M}) \textit{ into itself, such that } \| T^m - V \| < 1, \end{cases}$$

and under this condition they have obtained a remarkable result. In the present paper we shall develop this idea of N. Kryloff-N. Bogolioùboff and shall obtain a more precise result. This investigation will be carried out in §4, and our principal results may be summed up in the following two statements:

(1) *the condition* (D) *implies the condition* (K),

(2) *under the condition* (K), *all the results of W. Doeblin can be obtained even in a more precise form.*

Besides the *ergodic kernel* (= *ensemble final* due to W. Doeblin) we shall introduce a new notion of *ergodic part*, and it will be shown that this also plays a fundamental rôle in the investigation of the asymptotic behavior of the sequence $P^{(n)}(t, E)$ for large n.

In order to obtain these results, we shall develop in §§2 and 3 a general theory of iteration of bounded linear operations in Banach spaces. Although the

[5] (L) is, as usual, the space of all the (real- or complex-valued) measurable functions $x(t)$ which are absolutely integrable on Ω. For any $x(t) \in (L)$, its norm is defined by: $\| x \| = \int_\Omega | x(t) |\, dt$, and in (L), an integrable function $x(t)$ is called to be positive if we have $x(t) \geqq 0$ almost everywhere in t.

iteration of bounded linear operations in concrete Banach space (for example, in Hilbert space) was discussed by many authors, such a theory has not hitherto been developed in general Banach spaces. We shall show now that this is possible under general conditions, and the results thus obtained will find their full applications in §4.

Thus §§2 and 3 may be considered as preparations to §4. Nevertheless, these are also the principal chapters of this paper. Indeed, Theorem 1 (mean ergodic theorem) and Theorem 4 (uniform ergodic theorem) are among the main results of our paper. In order to explain the meaning of these theorems, let us recall the *mean ergodic theorem* of J. v. Neumann:

Let S be a one-to-one measure-preserving transformation of a space Ω into itself. Then

$$x \rightarrow T(x) = y: \qquad y(t) = x(S(t))$$

is a positive unitary transformation which maps the Hilbert space (L^2) (defined on Ω) into itself, and the mean ergodic theorem of J. v. Neumann says that, for any $x \in (L^2)$, the sequence $\{x_n\}$ $(n = 1, 2, \cdots)$, where $x_n = \frac{1}{n}(T + T^2 \cdots + T^n)x$, converges strongly to some element $\bar{x} \in (L^2)$.

This theorem may be generalized in two ways: firstly, (L^2) may be substituted by an arbitrary Banach space (B), and secondly, T may be substituted by an arbitrary quasi-weakly completely continuous[6] linear operation which maps the Banach space (B) into itself. The positiveness of T and the existence of the inverse operation T^{-1} are both not assumed. This theorem is called the *mean ergodic theorem in Banach spaces* and will be proved in §2.[7]

Moreover, if T is quasi-strongly completely continuous,[6] i.e., if T satisfies the condition (K) of N. Kryloff-N. Bogolioùboff, then the strong convergence in the mean ergodic theorem may be substituted by the uniform one. This is a generalization of the results of M. Fréchet [1], [2], C. Visser [1] and N. Kryloff-N. Bogolioùboff [1], [2], and will be called the *uniform ergodic theorem in Banach spaces*. We shall prove this in §3. This theorem is extremely powerful in the investigation of concrete Markoff's process in §4.

In concluding the introduction, we express our hearty thanks to Yukio Mimura and Shôkichi Iyanaga for their constant encouragements during the preparation of this paper.

CHAPTER 2. MEAN ERGODIC THEOREM IN BANACH SPACES[8]

(Generalization of J. v. Neumann's mean ergodic theorem)

In this chapter we are concerned with weakly completely continuous and quasi-weakly completely continuous linear operations in general (real or com-

[6] This notion will be explained at the beginning of §2.

[7] Recently, the generalizations of the mean ergodic theorem were given also by N. Wiener [1], [2], G. Birkhoff [2] and N. Dunford [1]. We shall discuss these results in §2. See the remarks after Theorem 1 of §2.

[8] K. Yosida [2], S. Kakutani [1], K. Yosida and S. Kakutani [1].

plex) Banach spaces. By definition, a bounded linear operation T, which maps a Banach space (B) into itself, is called to be *weakly completely continuous* if it maps the unit sphere $\| x \| \leqq 1$ of (B) on a weakly compact set of (B). More generally, T is called to be *quasi-weakly completely continuous*, if there exist an integer m and a weakly completely continuous linear operation V, which maps (B) into itself, such that $\| T^m - V \| < 1$.

THEOREM 1. *Let T be a bounded linear operation which maps a Banach space (B) into itself. Let us further assume that*

(2.1) *there exists a constant C such that $\| T^n \| \leqq C$ for $n = 1, 2, \cdots$, and that*

$$\text{(2.2)} \quad \begin{cases} \text{for any } x \in (B), \text{ the sequence } \{x_n\}\ (n = 1, 2, \cdots), \text{ where } x_n = \frac{1}{n}(T + \\ T^2 + \cdots + T^n)x, \text{ contains a subsequence which converges weakly to a} \\ \text{point } \bar{x} \in (B). \end{cases}$$

Under these assumptions,

(2.3) *the sequence $\{x_n\}$ $(n = 1, 2, \cdots)$ converges strongly to a point $\bar{x}$,*

and if

we denote by T_1 the operation $x \to \bar{x}$, then T_1 is a bounded linear operation which maps (B) into itself and

$$TT_1 = T_1T = T_1^2 = T_1, \quad \| T_1 \| \leqq C. \tag{2.4}$$

PROOF. Let x be an arbitrary point of (B). By (2.2), there exists a subsequence $\{x_{n_\nu}\}$ $(\nu = 1, 2, \cdots)$ of $\{x_n\}$ $(n = 1, 2, \cdots)$ which converges weakly to a point $\bar{x}$ of (B). We shall first prove that we have

$$T(\bar{x}) = \bar{x}. \tag{2.5}$$

This is an easy consequence of the relation:

$$\begin{aligned} &\| T(x_n) - x_n \| \\ &= \left\| \frac{1}{n}(T^2 + T^3 + \cdots + T^{n+1})x - \frac{1}{n}(T + T^2 + \cdots + T^n)x \right\| \\ &= \left\| \frac{1}{n}(T^{n+1} - T)x \right\| \leqq \frac{2C}{n} \| x \|, \end{aligned} \tag{2.6}$$

which is valid for $n = 1, 2, \cdots$. Indeed, putting $n = n_\nu$ in (2.6) and making ν tend to ∞, we have (2.5) at once; for $\{x_{n_\nu}\}$ and $\{T(x_{n_\nu})\}$ converge weakly to $\bar{x}$ and $T(\bar{x})$ respectively.

We shall next prove (2.3). For this purpose, decompose x into two parts: $x = \bar{x} + (x - \bar{x})$. Then we have, by (2.5),

$$x_n = \bar{x} + \frac{1}{n}(T + T^2 + \cdots + T^n)(x - \bar{x}), \tag{2.7}$$

and consequently, in order to prove (2.3), we have only to prove that the second term of the right hand side of (2.7) converges strongly to 0 as $n \to \infty$.

Let the range of the linear operation $I - T$ (I is the identical transformation) be R, and denote its closure (in the strong sense) by $\bar{R}$. Since R is a linear subspace of (B), $\bar{R}$ is also closed in the weak sense. Since the sequence

$$\left\{I - \frac{1}{n}(T + T^2 + \cdots + T^n)\right\} x$$
$$= (I - T)\left(I + \frac{n-1}{n} T + \frac{n-2}{n} T^2 + \cdots + \frac{1}{n} T^{n-1}\right) x$$

converges weakly to $x - \bar{x}$, and since the right hand side clearly belongs to R, $x - \bar{x}$ belongs to $\bar{R}$. Consequently, in order to prove (2.3), we have only to prove that

(2.8) $\quad \frac{1}{n}(T + T^2 + \cdots + T^n)x$ converges strongly to 0

for any $y \epsilon \bar{R}$. This may be performed in two steps:

1st case. $y \epsilon R$. Taking a point $z \epsilon (B)$ such that $y = z - T(z)$, we have

$$\left\| \frac{1}{n}(T + T^2 + \cdots + T^n)y \right\| = \left\| \frac{1}{n}(T + T^2 + \cdots + T^n)(z - T(z)) \right\|$$
$$= \left\| \frac{1}{n}(T - T^{n+1})z \right\| \leqq \frac{2C}{n} \| z \| \to 0.$$

2nd case. $y \epsilon \bar{R}$. For any $\epsilon > 0$ there exists a point $y' \epsilon R$ such that $\| y - y' \| < \epsilon$. For this y' we have

$$\left\| \frac{1}{n}(T + T^2 + \cdots + T^n)y \right\| \leqq \left\| \frac{1}{n}(T + T^2 + \cdots + T^n)y' \right\|$$
$$+ \left\| \frac{1}{n}(T + T^2 + \cdots + T^n)(y - y') \right\|$$
$$\leqq \left\| \frac{1}{n}(T + T^2 + \cdots + T^n)y' \right\| + C\epsilon.$$

Since the first term of the right hand side converges strongly to 0 as $n \to \infty$ (by the result obtained in the 1st case), we have

$$\overline{\lim_{n\to\infty}} \left\| \frac{1}{n}(T + T^2 + \cdots + T^n)y \right\| \leqq C\epsilon;$$

and, since $\epsilon > 0$ is arbitrary, we have (2.8). Thus (2.8) is proved for any $y \epsilon \bar{R}$ and hereby (2.3) is completely proved.

Let now T_1 be the transformation $x \to \bar{x}$. T_1 is clearly linear and bounded: $\| T_1 \| \leqq C$. Since (2.5) is always true, we have $TT_1 = T_1$. Thus $T^nT_1 = T_1$ and $\frac{1}{n}(T + T^2 + \cdots + T^n)T_1 = T_1$ for $n = 1, 2, \cdots$, and consequently $T_1^2 = T_1$. In order to prove the relation: $T_1T = T_1$, we have only to start from the inequality:

$$\left\| \frac{1}{n}(T + T^2 + \cdots + T^n)T(x) - \frac{1}{n}(T + T^2 + \cdots + T^n)x \right\| \leqq \frac{2C}{n} \| x \| \to 0.$$

Since the terms on the left hand side converge strongly to $T_1T(x)$ and $T_1(x)$ respectively, we have $T_1T(x) = T_1(x)$ for any $x \in (B)$.

Thus the proof of Theorem 1 is completed.

REMARK 1. T_1 is a projection operator which maps (B) on the proper space (B_1) of T belonging to the proper value 1.[9] That is, T_1 does not vanish identically if and only if 1 is a proper value of T, and $T_1(x) = x$ if and only if we have $T(x) = x$.

REMARK 2. In virtue of (2.1), (2.2) is surely satisfied if T is weakly completely continuous. Hence

COROLLARY 1. *Let T be a weakly completely continuous linear operation which maps a Banach space (B) into itself. If there exists a constant C such that $\| T^n \| \leqq C$ for $n = 1, 2, \cdots$, then the sequence of operations $\left\{\frac{1}{n}(T + T^2 + \cdots + T^n)\right\}$ $(n = 1, 2, \cdots)$ converges strongly to a bounded linear operation T_1 which maps (B) into itself and which satisfies (2.4); i.e., mean ergodic theorem is valid in (B).*

The conditions of this Corollary are clearly satisfied if (B) is a Hilbert space and if T is a unitary operation in it. (Since Hilbert space itself is locally weakly compact.) Thus Theorem 1 is, in two respects, a generalization of the mean ergodic theorem of J. v. Neumann [1], [2]. (See also E. Hopf [1].) In the first place, our theorem is true in general complex Banach space, while the proof of J. v. Neumann is valid only in the case of a Hilbert space. In the second place, we do not assume that T is isometric, and the existence of the inverse operation T^{-1} is not required. The positiveness of the operation T is also not required. We have only to assume that the conditions (2.1) and (2.2) are satisfied. The first condition (2.1) is always satisfied with $C = 1$ in the case of ergodic theorems and Markoff's processes; and the second one (2.2) is satisfied in every weakly compact Banach space (for example, in regular Banach space and especially in (L^p) $(p > 1)$) for any bounded linear operation T. Consequently, mean ergodic theorem holds true in (L^p) $(p > 1)$ under the trivial condition (2.1). This result was also obtained independently by F. Riesz [1]. On the contrary, in the Banach space $(L) = (L^1)$, the condition (2.2) is not always satisfied for an arbitrary bounded linear operation T. In order that a bounded subset S of (L) be weakly compact, it is (necessary and) sufficient that the functions belonging to this subset S are uniformly integrable, that is, that there exists for any $\epsilon > 0$ a positive number $\delta > 0$ such that $\mathrm{mes}(E) < \delta$ implies

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[9] It is to be noted that in general Banach space (B), there does not necessarily exist, for any closed linear subspace (B'), a projection operator (of norm 1 or of finite norm) which maps (B) in (B').

$\int_E |x(t)|\,dt < \epsilon$ for any $x(t) \in S$. This condition is surely satisfied if there exists a function $x_0(t) \in (L)$ such that $|x| \leqq x_0$ (that is, $|x(t)| \leqq x_0(t)$ almost everywhere in t) for any $x(t) \in S$. Hence we have

COROLLARY 2. *Let T be a bounded linear operation which maps the Banach space (L) into itself. If there exists a constant C such that $\| T^n \| \leqq C$ for $n = 1, 2, \cdots$, and if there exists, for any $x(t) \in (L)$, a function $x_0(t) \in (L)$ such that $T^n(x) \leqq x_0 \left(\text{or, more generally, } \frac{1}{n}(T + T^2 + \cdots + T^n)x \leqq x_0 \right)$ for $n = 1, 2, \cdots$, then for any $x(t) \in (L)$ the sequence $\left\{ \frac{1}{n}(T + T^2 + \cdots + T^n)x \right\}$ $(n = 1, 2, \cdots)$ converges strongly to some function $\bar{x}(t) \in (L)$; i.e., mean ergodic theorem is valid in (L).*

In order to prove this, we have only to notice that there exists also an x_0' such that $T^n(-x) \leqq x_0'$ i.e. $-x_0' \leqq T^n(x)$ for $n = 1, 2, \cdots$.

This theorem was proved by F. Riesz [2] and an analogous theorem was also stated by Garrett Birkhoff [1] for abstract (L)-spaces. The result of G. Birkhoff is more general in formulation, but it was pointed out that in essential the two theorems are equivalent.[10]

Moreover, if T is an integral operator with the probability density kernel $p(t, s)$:

$$x \rightarrow T(x) = y \colon y(s) = \int_\Omega x(t) p(t, s)\, dt$$

$$\left(p(t, s) \geqq 0 \quad \text{for any } t, s; \qquad \int_\Omega p(t, s)\, ds = 1 \quad \text{for any } t\right),$$

then a (necessary and) sufficient condition that T be weakly completely continuous is that $p(t, s)$ is uniformly (in t) integrable in s, that is, that there exists for any $\epsilon > 0$ a positive number $\delta > 0$ such that $\text{mes}(E) < \delta$ implies $\int_E p(t, s)\, ds < \epsilon$ for any t. This condition was investigated by J. L. Doob [1] without being noticed that the linear operation T becomes weakly completely continuous under this condition. We have once[11] treated this case of the condition of J. L. Doob as an application of the mean ergodic theorem. But, on looking precisely into the detail of the fact, we have found that, in this case, the linear operation T satisfies even the (in some sense stronger) condition (K) of N. Kryloff-N. Bogolioùboff, and that the uniform ergodic theorem is true in this case. (Indeed, the condition (K) of N. Kryloff-N. Bogolioùboff follows from the condition (D) of W. Doeblin, which is weaker than that of J. L. Doob.)[12]

REMARK 3. Recently Garrett Birkhoff [2] proved that in every uniformly

[10] S. Kakutani [2].
[11] K. Yosida and S. Kakutani [1].
[12] K. Yosida [3].

convex Banach space mean ergodic theorem is valid for any bounded linear operation of norm not exceeding unity. This is also a generalization of the mean ergodic theorem of J. v. Neumann, since Hilbert space is uniformly convex. But this result of G Birkhoff is contained in our Theorem 1; for, every uniformly convex Banach space is regular and consequently locally weakly compact.[13] (It is, however, to be remarked that the proof of G. Birkhoff distinguishes itself by its extreme simplicity.)

REMARK 4. We can also prove the theorem of the following type:

COROLLARY 3. *Let S and T be two bounded linear operations which map a Banach space (B) into itself, and assume that these are commutative between themselves: $ST = TS$. If there exists a constant C such that $\| T^n \| \leqq C$ and $\| S^n \| \leqq C$ for $n = 1, 2, \cdots$, and if, for any $x \in (B)$, the sequence $\{x_{mn}\}$ $(m, n = 1, 2, \cdots)$, where*

$$x_{mn} = \frac{1}{mn}(S + S^2 + \cdots + S^m)(T + T^2 + \cdots + T^n)x,$$

contains a subsequence $\{x_{m_\nu n_\nu}\}$ $(m_\nu \to \infty,\ n_\nu \to \infty)$ which converges weakly to a point $\bar{x}$ of (B), then we have $\lim_{m,n\to\infty} x_{mn} = \bar{x}$ strongly for any $x \in (B)$; and if we denote the transformation $x \to \bar{x}$ by U, then U is a bounded linear operation which maps (B) into itself and we have

$$US = SU = UT = TU = U^2 = U \quad \text{and} \quad \| U \| \leqq C.$$

PROOF. It will be easily seen (exactly as in the proof of Theorem 1) that we have $S(\bar{x}) = T(\bar{x}) = \bar{x}$. Hence we have only to prove that the sequence $\left\{\frac{1}{mn}(S + S^2 + \cdots + S^m)(T + T^2 + \cdots + T^n)(x - \bar{x})\right\}$ converges strongly to 0 as $m \to \infty$ and $n \to \infty$ simultaneously. In order to prove this, denote by R the set of all the points $z \in (B)$ of the form: $z = (x - S(x)) + (y - T(y))$ with $x, y \in (B)$. R is clearly a linear subspace of (B) and $x - \bar{x}$ belongs to its strong (= weak) closure $\bar{R}$; for, we have

$$\begin{aligned}
&\left\{I - \frac{1}{mn}(S + S^2 + \cdots + S^m)(T + T^2 + \cdots + T^n)\right\}x \\
&\qquad = \left\{I - \frac{1}{m}(S + S^2 + \cdots + S^m)\right\}x \\
&\qquad\qquad + \left\{\frac{1}{m}(S + S^2 + \cdots + S^m)\right. \\
&\qquad\qquad\qquad \left. - \frac{1}{mn}(S + S^2 + \cdots + S^m)(T + T^2 + \cdots + T^n)\right\}x
\end{aligned}$$

[13] S. Kakutani [3]. See also D. Milman [1]. B. J. Pettis [1].

$$= (I - S)\left(I + \frac{m-1}{m} S + \frac{m-2}{m} S^2 + \cdots + \frac{1}{m} S^{m-1}\right) x$$

$$+ \frac{1}{m} (S + S^2 + \cdots + S^m)(I - T)$$

$$\left(I + \frac{n-1}{n} T + \frac{n-2}{n} T^2 + \cdots + \frac{1}{n} T^{n-1}\right) x,$$

and the left hand side converges weakly to $x - \bar{x}$. Consequently, we have only to prove that $\left\{\frac{1}{mn} (S + S^2 + \cdots + S^m)(T + T^2 + \cdots + T^n) z\right\}$ converges strongly to 0 for any $z \in \bar{R}$. This is clear if $z \in R$, and the general case $z \in \bar{R}$ may be treated in exactly the same manner as in the proof of Theorem 1.

Mean ergodic theorem of this type was first discussed by N. Wiener [1], [2]. N. Wiener treated the case when the inverse operators S^{-1} and T^{-1} both exist, and has also obtained the individual ergodic theorem of G. D. Birkhoff's type. Recently N. Dunford [1] has also announced the theorem of the same kind.

THEOREM 2. *Under the same assumptions as in Theorem 1, we have: for any complex number λ of absolute value 1, there exists a bounded linear operation T_λ, which maps (B) into itself, such that*

(2.9) $\quad \frac{1}{n}\left(\frac{T}{\lambda} + \frac{T^2}{\lambda^2} + \cdots + \frac{T^n}{\lambda^n}\right)$ *converges strongly to* T_λ,

(2.10) $\quad T_\lambda T = T T_\lambda = \lambda T_\lambda, \quad T_\lambda^2 = T_\lambda, \quad \| T_\lambda \| \leqq C,$

(2.11) $\quad \lambda \neq \mu$ *implies* $T_\lambda T_\mu = 0$,

(2.12) $\quad T_\lambda \neq 0$ *if and only if* λ *is a proper value of* T.

In this case, T_λ is a projection operator which maps (B) on the proper space (B_λ) of T belonging to the proper value λ, and $T_\lambda(x) = x$ if and only if we have $T(x) = \lambda x$. If we further put $T' = T - \sum_{i=1}^{k} \lambda_i T_{\lambda_i}$ for any system $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$ of complex numbers of modulus 1 with $\lambda_i \neq \lambda_j$ $(i \neq j)$ and $T_{\lambda_i} \neq 0$, then we have

(2.13) $\quad T_{\lambda_i} T' = T' T_{\lambda_i} = 0 \quad$ *for* $\quad i = 1, 2, \cdots, k,$

(2.14) $\quad T T' = T' T = T'^2,$

(2.15) $\quad T^n = \sum_{i=1}^{k} \lambda_i^n T_{\lambda_i} + T'^n \quad$ *for* $n = 1, 2, \cdots,$

(2.16) $\quad \begin{cases} \lambda \textit{ is a proper value of } T' \textit{ if and only if it is a proper value of } T \textit{ and} \\ \lambda \neq \lambda_i \textit{ for } i = 1, 2, \cdots, k. \end{cases}$

PROOF. (2.9) is clear from Theorem 1 if we consider T/λ instead of T. From this (2.10) follows at once. To prove (2.11), multiply (2.9) by T_μ from the right. Then we have, by (2.10),

$$\frac{1}{n}\left(\frac{\mu}{\lambda} + \frac{\mu^2}{\lambda^2} + \cdots + \frac{\mu^n}{\lambda^n}\right) T_\mu \quad \text{converges strongly to } T_\lambda T_\mu.$$

Since, in case $\lambda \neq \mu$, the left hand side converges strongly (even uniformly) to a zero operator, we have (2.11). (2.12) is clear from the Remark 1 to Theorem 1. (2.13), (2.14) and (2.15) may also be proved by easy calculations. To prove lastly (2.16), let λ, $|\lambda| \neq 0$, be a proper value of T and assume that $\lambda \neq \lambda_i$ for $i = 1, 2, \cdots, k$. Then there exists a point $x_0 \in (B)$ $(x_0 \neq 0)$ such that $T(x_0) = \lambda x_0$. Therefore, multiplying the both sides by T_{λ_i} from the left, we have, by (2.10), $\lambda_i T_{\lambda_i}(x_0) = \lambda T_{\lambda_i}(x_0)$ and consequently (since $\lambda \neq \lambda_i$ we have $T_{\lambda_i}(x_0) = 0$ for $i = 1, 2, \cdots, k$, so that we have $\lambda x_0 = T(x_0) = \sum_{i=1}^{k} \lambda_i T_{\lambda_i}(x_0) + T'(x_0) = T'(x_0)$. Hence λ is a proper value of T'. Conversely, let λ be a proper value of T'. Then there exists a point $x_0 \in (B)$ $(x_0 \neq 0)$ such that $T'(x_0) = \lambda x_0$. Since we have $T(x_0) = T\left(\frac{1}{\lambda} T'(x_0)\right) = \frac{1}{\lambda} T T'(x_0) = \frac{1}{\lambda} T'^2(x_0)$ (by (2.14)) $= \lambda x_0$, λ is also a proper value of T. In order to prove that $\lambda \neq \lambda_i$ for $i = 1, 2, \cdots, k$, assume that $\lambda = \lambda_i$ for some i. Then we have

$$\frac{1}{n}\left(\frac{T}{\lambda_i} + \frac{T^2}{\lambda_i^2} + \cdots + \frac{T^n}{\lambda_i^n}\right) x_0 = x_0 \quad \text{for } n = 1, 2, \cdots,$$

and consequently (by (2.9)) $T_{\lambda_i}(x_0) = x_0 \neq 0$, which leads to a contradiction, since $T_{\lambda_i}(x_0) = T_{\lambda_i}\left(\frac{1}{\lambda} T'(x_0)\right) = \frac{1}{\lambda} T_{\lambda_i} T'(x_0) = 0$ by (2.13).
The proof of Theorem 2 is hereby completed.

THEOREM 3. *In Theorems 1 and 2, the condition (2.2) may be substituted by the condition that T is quasi-weakly completely continuous.*

PROOF. We have only to prove that, under the condition (2.1) and the assumption that T is quasi-weakly completely continuous, there exists for any $x \in (B)$ a subsequence $\{x_{n_\nu}\}$ $(\nu = 1, 2, \cdots)$ of $\{x_n\}$ $(n = 1, 2, \cdots)$ which converges weakly to a point $\bar{x} \in (B)$. In order to prove this, put $T^m = V + D$ with $\|D\| = \alpha < 1$. Then we have

$$T^{pm} = V_p + D^p, \tag{2.17}$$

where $V_p = T^{pm} - (T^m - V)^p$ is weakly completely continuous with $V_1 = V$ and $\|D^p\| \leqq \alpha^p$.[14] Hence, putting

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[14] If V and W are weakly completely continuous, then $V + W$, TV and VT are also weakly completely continuous for any bounded linear operation T. Thus the totality of all the weakly completely continuous linear operations constitutes an ideal in the ring of all bounded linear operations. Expanding $T^{pm} - (T^m - V)^p$, the term T^{pm} vanishes and there remain only those terms (finite in number) which contain at least one V-factor. Consequently V_p is weakly completely continuous.

It is to be remarked that the same is also true for strongly completely continuous linear operations.

$$x_{n,m} = \frac{1}{n}(T + T^2 + \cdots + T^m)x$$

for brevity's sake, we have for $n > pm$

$$x_n = \frac{1}{n}(T + T^2 + \cdots + T^n)x$$

$$= \frac{1}{n}(T + T^2 + \cdots + T^{pm})x + \frac{1}{n}T^{pm}(T + T^2 + \cdots + T^{n-pm})x$$

$$= x_{n,pm} + T^{pm}(x_{n,n-pm}) = x_{n,pm} + V_p(x_{n,n-pm}) + D^p(x_{n,n-pm}).$$

Since $\| x_{n,n-pm} \| \leqq C \cdot \| x \|$ for any $n > pm$ (by (2.1)), there exists, for any p, a subsequence $\{n_\nu\}$ $(\nu = 1, 2, \cdots)$ of $\{n\}$ $(n = 1, 2, \ldots)$ such that $\{V_p(x_{n_\nu, n_\nu - pm})\}$ $(\nu = 1, 2, \cdots)$ converges weakly to a point $\bar{x}_p \in (B)$. Consequently we have

$$\overline{\lim_{\nu\to\infty}} | f(x_{n_\nu}) - f(\bar{x}_p) | \leqq \lim_{\nu\to\infty} | f(x_{n_\nu, pm}) | + \overline{\lim_{\nu\to\infty}} | f(D^p(x_{n_\nu, n_\nu - pm})) |$$

for any bounded linear functional $f(x)$ defined on (B). Since

$$| f(x_{n_\nu, pm}) | \leqq \| f \| \cdot \| x_{n_\nu, pm} \| \leqq \| f \| \cdot \frac{pm}{n_\nu} \cdot C \| x \| \to 0,$$

and

$$| f(D^p(x_{n_\nu, n_\nu - pm})) | \leqq \| f \| \cdot \| D^p \| \cdot \| x_{n_\nu, n_\nu - pm} \| \leqq \| f \| \cdot \alpha^p \cdot C \cdot \| x \|,$$

we have

$$\overline{\lim_{\nu\to\infty}} | f(x_{n_\nu}) - f(\bar{x}_p) | \leqq \| f \| \cdot \alpha^p \cdot C \| x \| \tag{2.18}$$

for any bounded linear functional $f(x)$ defined on (B).

Applying the diagonal method, we may assume that (2.18) holds for any bounded linear functional $f(x)$ defined on (B) and for $p = 1, 2, \cdots$. Of course, $\bar{x}_p$ may depend of p. Let us now consider the sequence $\{\bar{x}_p\}$ $(p = 1, 2, \cdots)$. From (2.18) we have

$$| f(\bar{x}_p) - f(\bar{x}_q) | \leqq (\alpha^p + \alpha^q) \cdot \| f \| C \| x \|,$$

and since $f(x)$ is an arbitrary bounded linear functional defined on (B), we have $\| \bar{x}_p - \bar{x}_q \| \leqq (\alpha^p + \alpha^q) \cdot C \cdot \| x \|$ for any p and q. Consequently, (since $\alpha < 1$), $\{\bar{x}_p\}$ $(p = 1, 2, \cdots)$ is a fundamental sequence in (B). If we put $\bar{x} = \lim_{p\to\infty} \bar{x}_p$, then we have, for any p, (from (2.18))

$$\overline{\lim_{\nu\to\infty}} | f(x_{n_\nu}) - f(\bar{x}) | \leqq \overline{\lim_{\nu\to\infty}} | f(x_{n_\nu}) - f(\bar{x}_p) | + | f(\bar{x}_p) - f(\bar{x}) |$$

$$\leqq \| f \| \cdot \alpha^p \cdot C \| x \| + \| f \| \cdot \| \bar{x}_p - \bar{x} \|.$$

Since p is arbitrary and since the right hand side tends to 0 as $p \to \infty$, we have $\lim_{\nu\to\infty} | f(x_{n_\nu}) - f(\bar{x}) | = 0$. Since $f(x)$ is an arbitrary bounded linear functional defined on (B), we have thus proved that the sequence $\{x_{n_\nu}\}$ $(\nu = 1, 2, \cdots)$ converges weakly to $\bar{x}$. The proof of Theorem 3 is hereby completed.

REMARK. All what we have obtained in Theorems 1 and 2 is also valid for quasi-weakly completely continuous linear operations. We can also prove several propositions concerning the conjugate operators of such bounded linear operations.[15] But we shall not go into the detail, which is not essential in the further discussions of this paper, and we shall proceed to the case of strongly completely continuous and quasi-strongly completely continuous linear operations.

CHAPTER 3. UNIFORM ERGODIC THEOREM IN BANACH SPACES[16]

(Generalization of the theorems of Fréchet-Kryloff-Bogolioùboff)

In this chapter we shall discuss strongly completely continuous and quasi-strongly completely continuous linear operations. By definition, a bounded linear operation T of a Banach space (B) into itself is called to be *strongly completely continuous* if it maps the unit sphere $\| x \| \leqq 1$ of (B) on a strongly compact set of (B). More generally, T is called to be *quasi-strongly completely continuous,* if there exist an integer m and a strongly completely continuous linear operation V, which maps (B) into itself, such that $\| T^m - V \| < 1$.

Since every strongly completely continuous linear operation is a fortiori weakly completely continuous, all what we have obtained in §2 is also valid for the case of strongly completely continuous or quasi-strongly completely continuous linear operations. Moreover, we can show that in the present case, the *strong* convergence in Theorems 1, 2 and 3 may be substituted by the *uniform* one. These are, as will be seen from the proofs below, the consequences of a theorem of F. Riesz and its generalizations to the case of quasi-strongly completely continuous linear operations (Lemmas 3.1 and 3.2).

THEOREM 4. *Let T be a strongly completely continuous or quasi-strongly completely continuous linear operation which maps a Banach space (B) into itself. If there exists a constant C such that $\| T^n \| \leqq C$ for $n = 1, 2, \cdots$, then the proper values λ of T of modulus 1 (if such proper value ever exists) are finite in number and each of them is of finite multiplicity. Let us denote these by $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then there exists a system of strongly completely continuous linear operations $T_{\lambda_1}, T_{\lambda_2}, \cdots, T_{\lambda_k}$ (each of them $\neq 0$), and a strongly completely continuous or quasi-strongly completely continuous linear operation S (which might vanish), such that T^n is decomposed into the form:*

$$T^n = \sum_{i=1}^{k} \lambda_i^n T_{\lambda_i} + S^n, \qquad n = 1, 2, \cdots, \tag{3.1}$$

[15] K. Yosida and S. Kakutani [1], K. Yosida [4].

[16] K. Yosida [1], [4], S. Kakutani [1].

with

$$(3.2)\qquad \begin{cases} TT_{\lambda_i} = T_{\lambda_i}T = \lambda_i T_{\lambda_i}, & T_{\lambda_i}^2 = T_{\lambda_i}, \quad T_{\lambda_i}T_{\lambda_j} = 0 \quad (i \neq j), \\ T_{\lambda_i}S = ST_{\lambda_i} = 0, & \| T_{\lambda_i} \| \leqq C, \quad i = 1, 2, \cdots, k, \end{cases}$$

and

$$(3.3)\qquad \| S^n \| \leqq \frac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots,$$

where M and ϵ are positive constants which are independent of n.

REMARK. Theorem 4 is a generalization of the results of C. Visser, M. Fréchet and N. Kryloff-N. Bogolioùboff. C. Visser [1] discussed the iteration of strongly completely continuous linear operations in Hilbert space, and has obtained only strong convergence. M. Fréchet [1], [2] discussed the case of an integral operator T with bounded density kernel $p(t, s)$:

$$x \to T(x) = y: \quad y(s) = \int_\Omega x(t)p(t, s)\, dt,$$

and has obtained a considerably precise result after somewhat long calculations. It was, however, shown by Y. Mimura and the authors (Lemma 4.5 in §4) that, for such integral operation T with bounded density kernel, the second iterate T^2 already becomes strongly completely continuous as an operator which maps the Banach space (L) into itself. Hence the theorem of M. Fréchet may be considered as a special case of Theorem 4. N. Kryloff-N. Bogolioùboff [1], [2] treated the case of an integral operator with general probability kernel $P(t, E)$, which maps the Banach space $(\mathbf{M})$ into itself:

$$x \to T(x) = y: \quad y(E) = \int_\Omega x(dt)P(t, E),$$

and have obtained the same results (they announced these without proof) under the same conditions as in Theorem 4. It is, however, to be noted, that Theorem 4 holds true even for the general case when the Banach space (B) is arbitrary and when the positiveness of the operation T is not assumed. The more detailed discussions of such operations and their iterations, where the condition of positiveness plays its essential rôle, will be fully developed in §4.

In order to prove Theorem 4, we shall first prove two Lemmas which may be considered as generalizations of the results of F. Riesz [1].

LEMMA 3.1. *If T is quasi-strongly completely continuous, then the proper values λ of T do not accumulate to a point not interior to the unit circle $|\lambda| = 1$ of the complex plane.*

PROOF. Take an integer p so large that we have $\alpha^p < \frac{1}{4}$ where $T^m = V + D$, $\| D \| = \alpha < 1$. We have (exactly as we have obtained (2.17) in §2)

$$(3.4)\qquad T^{pm} = V_p + D^p, \qquad \| D^p \| \leqq \alpha^p < \tfrac{1}{4},$$

where V_p is a strongly completely continuous linear operation which maps (B) into itself. From this it will be easily seen that we have only to consider the case when $m = 1$ and $\alpha < \frac{1}{4}$.

In order to prove Lemma 3.1 in this case, let $\{\lambda_n\}$ $(n = 1, 2, \cdots, \lambda_n \neq \lambda_m$ $(n \neq m))$ be a sequence of proper values of T which converges to a point λ_0 of modulus not smaller than 1. We shall deduce a contradiction from these assumptions. For this purpose, let $\{x_n\}$ $(n = 1, 2, \cdots)$ be a proper element of T corresponding to the proper value λ_n : $T(x_n) = \lambda_n x_n$, $x_n \neq 0$, $n = 1, 2, \cdots$. We shall first prove that these x_n are mutually linearly independent. This may be done by mathematical induction. Let $x_1, x_2, \cdots, x_{n-1}$ be already linearly independent and assume that x_n depends linearly on $x_1, x_2, \cdots, x_{n-1}$: $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$. Then we have

$$\sum_{i=1}^{n-1} \alpha_i(\lambda_n - \lambda_i)x_i = \lambda_n x_n - \sum_{i=1}^{n-1} \alpha_i \lambda_i x_i = T\left(x_n - \sum_{i=1}^{n-1} \alpha_i x_i\right) = 0.$$

Since $\{\alpha_i\}$ $(i = 1, 2, \cdots, n - 1)$ do not vanish simultaneously, this is a contradiction to the assumption that $x_1, x_2, \cdots, x_{n-1}$ are mutually linearly independent.

Thus we have proved that x_n $(n = 1, 2, \cdots)$ are mutually linearly independent. Consequently, the $(n - 1)$-dimensional subspace (X_{n-1}) of (B), which is spanned by $x_1, x_2, \cdots, x_{n-1}$, is a true subspace of the linear space (X_n) which is spanned by $x_1, x_2, \cdots, x_n$. Hence, by a well-known theorem of F. Riesz [1], there exists a sequence of points $\{y_n\}$ $(n = 1, 2, \cdots)$ such that $y_n \in (X_n)$, $\| y_n \| = 1$ and $\| y_n - x \| > \frac{1}{2}$ for any $x \in (X_{n-1})$. Since each y_n is of the form: $y_n = \sum_{i=1}^{n} \alpha_i x_i$, we have

$$y_n - T\left(\frac{y_n}{\lambda_n}\right) = \sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{n} \frac{\alpha_i}{\lambda_n} T(x_i) = \sum_{i=1}^{n-1} \alpha_i \left(1 - \frac{\lambda_i}{\lambda_n}\right) x_i \in (X_{n-1})$$

and

$$T\left(\frac{y_m}{\lambda_m}\right) = \sum_{i=1}^{m} \frac{\alpha_i}{\lambda_m} T(x_i) = \sum_{i=1}^{m} \alpha_i \frac{\lambda_i}{\lambda_m} x_i \in (X_m)$$

for any n and m. Consequently, the trivial identity:

$$T\left(\frac{y_n}{\lambda_n}\right) - T\left(\frac{y_m}{\lambda_m}\right) = y_n - \left\{\left(y_n - T\left(\frac{y_n}{\lambda_n}\right)\right) + T\left(\frac{y_m}{\lambda_m}\right)\right\}$$

implies

$$\left\| T\left(\frac{y_n}{\lambda_n}\right) - T\left(\frac{y_m}{\lambda_m}\right) \right\| > \tfrac{1}{2} \quad \text{for} \quad n > m. \tag{3.5}$$

On the other hand, since V is strongly completely continuous and since the sequence $\left\{\frac{y_n}{\lambda_n}\right\}$ $(n = 1, 2, \cdots)$ is uniformly bounded, there exist two sequences of integers $\{n_\nu\}$ and $\{m_\nu\}$ $(n_\nu > m_\nu, \nu = 1, 2, \cdots)$ such that $\lim_{\nu\to\infty} \left\| V\left(\frac{y_{n_\nu}}{\lambda_{n_\nu}}\right) - V\left(\frac{y_{m_\nu}}{\lambda_{m_\nu}}\right) \right\| = 0$. This is, however, a contradiction with (3.5), since we have by assumption

$$\left\| \left\{ T\left(\frac{y_{n_\nu}}{\lambda_{n_\nu}}\right) - T\left(\frac{y_{m_\nu}}{\lambda_{m_\nu}}\right) \right\} - \left\{ V\left(\frac{y_{n_\nu}}{\lambda_{n_\nu}}\right) - V\left(\frac{y_{m_\nu}}{\lambda_{m_\nu}}\right) \right\} \right\|$$
$$\leqq \left\| T\left(\frac{y_{n_\nu}}{\lambda_{n_\nu}}\right) - V\left(\frac{y_{n_\nu}}{\lambda_{n_\nu}}\right) \right\| + \left\| T\left(\frac{y_{m_\nu}}{\lambda_{m_\nu}}\right) - V\left(\frac{y_{m_\nu}}{\lambda_{m_\nu}}\right) \right\|$$
$$= \left\| D\left(\frac{y_{n_\nu}}{\lambda_{n_\nu}}\right) \right\| + \left\| D\left(\frac{y_{m_\nu}}{\lambda_{m_\nu}}\right) \right\| \leqq \alpha\left(\left\| \frac{y_{n_\nu}}{\lambda_{n_\nu}} \right\| + \left\| \frac{y_{m_\nu}}{\lambda_{m_\nu}} \right\| \right) \rightarrow \frac{2\alpha}{|\lambda_0|} < \tfrac{1}{2}.$$

Thus Lemma 3.1 is completely proved.

LEMMA 3.2. *If T is quasi-strongly completely continuous, then the proper space (B_λ) of T belonging to a proper value λ of T of modulus 1 is of finite dimension.*

PROOF. This may be done in exactly the same manner as in the proof of Lemma 3.1. We may again assume that $m = 1$ and $\alpha < \frac{1}{4}$.

If, for some proper value λ with $|\lambda| = 1$, the proper space (B_λ) is not of finite dimension, then there exists a sequence of points $x_n \in (B_\lambda)$ such that $\| x_n \| = 1$, $T(x_n) = \lambda x_n (n = 1, 2, \cdots)$ and $\| x_n - x_m \| > \frac{1}{2}$ for $n > m$.

On the other hand, since V is strongly completely continuous, there exist two sequences $\{n_\nu\}$ and $\{m_\nu\}$ $(n_\nu > m_\nu, \nu = 1, 2, \cdots)$ such that $\lim_{\nu\to\infty} \| V(x_{n_\nu}) - V(x_{m_\nu}) \| = 0$. This will, however, lead to a contradiction since we have

$$\begin{aligned}\tfrac{1}{2} < \| x_{n_\nu} - x_{m_\nu} \| &= \| \lambda(x_{n_\nu} - x_{m_\nu}) \| = \| T(x_{n_\nu}) - T(x_{m_\nu}) \| \\ &\leqq \| V(x_{n_\nu}) - V(x_{m_\nu}) \| + \| D(x_{n_\nu}) - D(x_{m_\nu}) \| \\ &\leqq \| V(x_{n_\nu}) - V(x_{m_\nu}) \| + 2\alpha.\end{aligned}$$

The proof of Lemma 3.2 is hereby completed.

PROOF OF THEOREM 4. Now, in order to prove Theorem 4, let us recall that Theorem 3 is valid in our case. From Lemma 3.1, there exists only a finite number of proper values λ of T on the unit circle $|\lambda| = 1$. Let us denote these proper values by $\lambda_1, \lambda_2, \cdots, \lambda_k$. Then there exists (by Theorem 2) for each λ_i a bounded linear operation T_{λ_i}, which maps (B) on the proper space (B_{λ_i}) of T belonging to the proper value λ_i. T_{λ_i} satisfies all the conditions of Theorem 2 and is strongly completely continuous since each (B_{λ_i}) is of finite dimension by Lemma 3.2. Let us put

$$S = T - \sum_{i=1}^{k} \lambda_i T_{\lambda_i} \tag{3.6}$$

and consider the bounded linear operation S thus defined. We shall prove that this S is also quasi-strongly completely continuous and that its iterations S^n are uniformly bounded. This is an easy consequence of the relation $S^n = T^n - \sum_{i=1}^{k} \lambda_i^n T_{\lambda_i}$, which follows immediately from (3.6) and the result of Theorem 2. Indeed, there exists a constant $C' \equiv C + \sum_{i=1}^{k} \| T_{\lambda_i} \| \leqq (k + 1)C$ such that $\| S^n \| \leqq \| T^n \| + \sum_{i=1}^{k} \| T_{\lambda_i} \| \leqq C'$ for $n = 1, 2, \cdots$, and also an integer m and a strongly completely continuous linear operation $V' \equiv V - \sum_{i=1}^{k} \lambda_i^m T_{\lambda_i}$ such that $\| S^m - V' \| = \| T^m - V \| < 1$.

Moreover, S has no proper value of modulus 1, so that in order to complete the proof of Theorem 4, we have only to prove the following

LEMMA 3.3. *If, in addition to the assumptions in Theorem* 4, *T has no proper value of modulus* 1, *then there exist two constants M and $\epsilon > 0$ such that*

$$\| T^n \| \leqq \frac{M}{(1+\epsilon)^n} \quad \textit{for} \quad n = 1, 2, \cdots. \tag{3.7}$$

PROOF. It is sufficient to prove the case $m = 1$. For, if Lemma 3.3 is true for the case $m = 1$, then there exist two constants M' and $\epsilon' > 0$ such that $\| T^{mn} \| \leqq M'/(1 + \epsilon')^n$ for $n = 1, 2, \cdots$, and it is easy to deduce from this the existence of two constants M and $\epsilon > 0$ which satisfy (3.7).

Now, in order to prove Lemma 3.3 in the case $m = 1$, put $T = V + D$, where V is strongly completely continuous and $\| D \| = \alpha < 1$. By a well-known result, $I - \lambda D$ (I is the identical transformation) admits a unique inverse $I + \lambda D(\lambda) \equiv I + \sum_{n=1}^{\infty} \lambda^n D^n$ which is regular in $| \lambda | < \frac{1}{\alpha}$, and we have $(I + \lambda D(\lambda))(I - \lambda T) = (I + \lambda D(\lambda))(I - \lambda V - \lambda D) = I - \lambda V - \lambda^2 D(\lambda) V \equiv I - V(\lambda)$, where $V(\lambda) \equiv \lambda V + \lambda^2 D(\lambda) V$ is regular in λ for $| \lambda | < \frac{1}{\alpha}$, and is strongly completely continuous for each λ with $| \lambda | < \frac{1}{\alpha}$. By Lemma 3.1, there exists a positive number $\eta > 0$ such that T has no proper value λ in $1 - \eta < | \lambda | < 1 + \eta$. Put $2\epsilon = \min\left(\frac{1}{\alpha} - 1, \eta\right)$ and consider the domain Δ: $1 - 2\epsilon < | \lambda | < 1 + 2\epsilon$. The equation $(I - V(\lambda))x = 0$ has no non-trivial solution $x \neq 0$ for each $\lambda \in \Delta$. For, if there exists an $x_0 \neq 0$ with $(I - V(\lambda))x_0 = 0$, then we have $(I - \lambda T)x_0 = (I - \lambda D)(I - V(\lambda))x_0 = 0$, and this is a contradiction since T has no proper value in Δ. Consequently, since $V(\lambda)$ is strongly completely continuous for any $\lambda \in \Delta$, there exists, by a theorem of F. Riesz, a unique inverse $I - K(\lambda)$ of $I - V(\lambda)$ for any $\lambda \in \Delta$; and it will be easily seen that $I - \lambda R(\lambda) = (I - K(\lambda))(I + \lambda D(\lambda))$ is an inverse of $I - \lambda T$ for each $\lambda \in \Delta$.

Thus we have proved that $I - \lambda T$ has an inverse for any $\lambda \in \Delta$. Since it is clear from the uniform boundedness of $\{T^n\}$ $(n = 1, 2, \cdots)$ that $I - \lambda T$ has an inverse $I + \sum_{n=1}^{\infty} \lambda^n T^n$ for any λ with $| \lambda | < 1$, we have thus proved the existence of an inverse $(I - \lambda T)^{-1}$ for any λ with $| \lambda | < 1 + 2\epsilon$. Consequently, by a theorem of M. Nagumo [1], $(I - \lambda T)^{-1}$ is regular in λ for $| \lambda | < 1 + 2\epsilon$ and the series of C. Neumann: $(I - \lambda T)^{-1} = I + \sum_{n=1}^{\infty} \lambda^n T^n$ converges in the uniform sense in $| \lambda | < 1 + 2\epsilon$. Hence there exists a constant M such that the inequality (3.7) is valid for $n = 1, 2, \cdots$.

The proof of Lemma 3.3 and herewith the proof of Theorem 4 are completed.

COROLLARY. *Under the same assumptions as in Theorem* 4 *we have*:

(i) *For any complex number λ with $| \lambda | = 1$, there exists a strongly completely continuous linear operation T_λ, which maps (B) into itself, such that*

$$\left\| \frac{1}{n}\left(\frac{T}{\lambda} + \frac{T^2}{\lambda^2} + \cdots + \frac{T^n}{\lambda^n}\right) - T_\lambda \right\| \leqq \frac{M}{n} \quad \text{for} \quad n = 1, 2, \cdots,$$

where M is a constant which is independent of n, and T_λ does not vanish identically if and only if λ is a proper value of T.

(ii) *In order that the sequence $\{T^n\}$ $(n = 1, 2, \cdots)$ converge in the uniform sense to a zero operation, it is necessary and sufficient that T has no proper value of absolute value* 1.

(iii) *In order that the sequence $\{T^n\}$ $(n = 1, 2, \cdots)$ converge in the uniform sense to a bounded linear operation $T_1 \neq 0$, it is necessary and sufficient that* 1 *is a proper value of T and that T has no other proper value of absolute value* 1.

In (ii) *and in* (iii), *if the sequence $\{T^n\}$ $(n = 1, 2, \cdots)$ converges in the uniform sense, then it is of the order of geometrical progression; that is, there exist a constant M and a positive number ϵ both independent of n, such that we have $\| T^n \| \leqq M/(1 + \epsilon)^n$ and $\| T^n - T_1 \| \leqq M/(1 + \epsilon)^n$ respectively for $n = 1, 2, \cdots$.*

Chapter 4. Markoff's Process[17]

§4.1. Introduction. Let us denote by $P(t, E)$ the transition probability that a point t of the unit interval $\Omega = (0, 1)$ is transferred, by a simple Markoff's process, into a Borel set E of Ω after the elapse of a unit-time. We have always $P(t, E) \geqq 0$ and $P(t, \Omega) = 1$. We shall assume that $P(t, E)$ is completely additive for Borel sets E if t is fixed, and that $P(t, E)$ is Borel measurable in t if E is fixed. Then the transition probability $P^{(n)}(t, E)$ that a point $t \in \Omega$ is transferred into a Borel set E of Ω after the elapse of n unit-times is given recurrently by

$$(4.1) \qquad \begin{aligned} P^{(n)}(t, E) &= \int_\Omega P^{(n-1)}(t, ds)P(s, E) = \int_\Omega P(t, ds)P^{(n-1)}(s, E), \\ & \qquad\qquad\qquad\qquad n = 2, 3, \cdots, \\ P^{(1)}(t, E) &= P(t, E), \end{aligned}$$

where the integration is of Radon-Stieltjes type.

Consider the complex Banach space **(M)** of the complex-valued completely additive set functions $x(E)$ defined for all Borel set E of Ω. For any $x(E) \in$ **(M)**, its norm is defined by: $\| x \| =$ total variation of $| x(E) |$ on Ω. Then we have

Lemma 4.1. *The integral operator*

$$(4.2) \qquad x \to T(x) = y: \quad y(E) = \int_\Omega x(dt)P(t, E)$$

is a bounded linear operation which maps the Banach space **(M)** *into itself and* $\| T \| = 1$.

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[17] K. Yosida and S. Kakutani [1], K. Yosida [3], S. Kakutani [4].

On the other hand, if we consider the complex Banach space (M^*) of all the complex-valued bounded Borel measurable functions $x(t)$ defined on Ω, with $\| x \| = \text{l.u.b.}_{t \in \Omega} | x(t) |$ as its norm, then we have

LEMMA 4.2. *The integral operator*

$$x \rightarrow \overline{T}(x) = y: \quad y(t) = \int_\Omega P(t, ds)x(s) \tag{4.3}$$

is a bounded linear operation which maps the Banach space (M^*) into itself and $\| T \| = 1$.

LEMMA 4.3. *For any $x(E) \in (\mathbf{M})$ and $y(t) \in (M^*)$, we have*

$$\int_\Omega x(dt) \left(\int_\Omega P(t, ds)y(s) \right) = \int_\Omega x_1(ds)y(s), \tag{4.4}$$

where $x_1(E) = \int_\Omega x(dt)P(t, E)$.

These three Lemmas are clear from the properties of $P(t, E)$.

REMARK. By virtue of Lemmas 4.1 and 4.2, $P^{(n)}(t, E)$ can be defined recurrently by (4.1). Hence $P^{(n)}(t, E)$ is completely additive for Borel sets E if t is fixed, and $P^{(n)}(t, E)$ is Borel measurable in t if E is fixed. Clearly we have

$$P^{(n)}(t, E) \geqq 0, \qquad P^{(n)}(t, \Omega) = 1, \qquad n = 1, 2, \cdots. \tag{4.5}$$

Moreover, by the repeated use of Lemma 4.3, we have

$$P^{(m+n)}(t, E) = \int_\Omega P^{(m)}(t, ds)P^{(n)}(s, E) \tag{4.6}$$

for any m and n, and it will be easily seen that the iterated operators T^n and $\overline{T}^n$ are defined by the kernel $P^{(n)}(t, E)$. We have clearly

$$\| T^n \| = \| \overline{T}^n \| = 1, \qquad n = 1, 2, \cdots. \tag{4.7}$$

It is now the purpose of this chapter to investigate the asymptotic behavior of the sequence $\{P^{(n)}(t, E)\}$ for large n. We shall treat this problem by considering $P^{(n)}(t, E)$ as the kernel of the integral operators T^n and $\overline{T}^n$. Since the Banach spaces $(\mathbf{M})$ and (M^*) are not conjugate to each other, these two operators T^n and $\overline{T}^n$ are not the conjugate operators to each other in the strict sense which was given by S. Banach [1]. But in essential, these play the same rôle.

Our problem is not quite easy if we have no further assumptions on the kernel $P(t, E)$. Our fundamental assumptions are the conditions (D) and (K) which were stated in §1. The first condition (D), which is due to W. Doeblin [1] is more general than those given by B. Hostinsky, M. Fréchet and J. L. Doob. The second one (K) is due to N. Kryloff-N. Bogolioùboff [1], [2] and was introduced by them independently of W. Doeblin. We shall show (§4.7) that the condition (D) implies (K), and under the condition (K) all the results of W. Doeblin will be obtained in a more precise form (§§4.2–4.6). Our principal

results are stated in Theorems 5–12. Theorem 5 is a restatement of the uniform ergodic theorem (Theorem 4 of §3), and this is a starting point of all the discussions of this chapter. Among other theorems, Theorem 6 is to be noticed. The formula (4.23) will show how the notions of the Banach spaces (**M**) and (M^*) are essential in our problems. As a corollary to Theorem 6, we shall obtain the new notion of ergodic parts (Theorem 7); and the decomposition of Ω into ergodic kernels (= "ensembles finals" of W. Doeblin) and the dissipative part (Theorem 8) is also a direct consequence of Theorem 6. Moreover, using the fact that under the condition (K) each proper value of T of modulus 1 is a root of unity (Theorem 9), the subdivision of the ergodic parts (and ergodic kernels) into cyclic parts will be easily obtained (Theorem 11 and its Corollary).

The classical results concerning the Markoff's process with a finite number of possible states may be easily obtained from our Theorems. In order to obtain these results, we have only to take a kernel $P(t, E)$ of the special type. This will be carried out in §4.8. In this way, the hitherto known results concerning the Markoff's process with a finite number or a continuum of possible states are obtained in a more precise form by a unified method.

§4.2. Spectral decomposition of $P^{(n)}(t, E)$ under the condition (K).

Theorem 5. *Under the condition* (K), $P^{(n)}(t, E)$ *is decomposed into the form*:

$$P^{(n)}(t, E) = \sum_{i=1}^{k} \lambda_i^n P_{\lambda_i}(t, E) + S^{(n)}(t, E), \qquad n = 1, 2, \cdots, \tag{4.8}$$

where $\{\lambda_i\}$ $(i = 1, 2, \cdots, k)$ *are the proper values of* T *of modulus* 1, *and*

$$\underset{t \in \Omega, E \subset \Omega}{\text{l.u.b.}} \left| \frac{1}{n} \sum_{m=1}^{n} \frac{P^{(m)}(t, E)}{\lambda_i^m} - P_{\lambda_i}(t, E) \right| \leqq \frac{M}{n}, \tag{4.9}$$

$$\int_\Omega P^{(n)}(t, ds) P_{\lambda_i}(s, E) = \int_\Omega P_{\lambda_i}(t, ds) P^{(n)}(s, E) = \lambda_i^n P_{\lambda_i}(t, E), \tag{4.10}$$

$$\int_\Omega P_{\lambda_i}(t, ds) P_{\lambda_j}(s, E) = P_{\lambda_i}(t, E) \quad \textit{or } 0 \textit{ according as } i = j \textit{ or } i \neq j, \tag{4.11}$$

$$\int_\Omega P_{\lambda_i}(t, ds) S(s, E) = \int_\Omega S(t, ds) P_{\lambda_i}(s, E) = 0, \tag{4.12}$$

$$\underset{t \in \Omega, E \subset \Omega}{\text{l.u.b.}} \; | S^{(n)}(t, E) | \leqq \frac{M}{(1 + \epsilon)^n}, \tag{4.13}$$

$i = 1, 2, \cdots, k$; $n = 1, 2, \cdots$, *where* M *and* ϵ *are positive constants which are independent of* n.

Proof. This theorem follows directly from Theorem 4 (uniform ergodic theorem) of §3. We have only to notice that, by Theorem 4, only the decomposition of the operators T^n is given and that the decomposition of the kernels

$P^{(n)}(t, E)$ is not yet obtained. But, since the convergence $\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} T^m/\lambda_i^m =$ T_{λ_i} is uniform, the decomposition of the kernels is simultaneously obtained.

Remark. The uniform limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \frac{P^{(m)}(t, E)}{\lambda^m} = P_\lambda(t, E)$$

exists for any λ with $|\lambda| = 1$. By putting $\lambda = 1$ and remembering (4.5), we see that $P_1(t, E)$ is not identically zero. Hence $\lambda = 1$ is a proper value of T. If we put $\lambda_1 = 1$, then the integral operator T_1 defined by the kernel

$$P_1(t, E) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t, E) \qquad \text{(uniform limit)} \tag{4.14}$$

is a projection operator, which maps the Banach space (**M**) on the proper space of T belonging to the proper value 1. More precisely, we have

$$\left\| \frac{T + T^2 + \cdots + T^n}{n} - T_1 \right\| \leqq \frac{M}{n}, \qquad n = 1, 2, \cdots, \tag{4.15}$$

with a positive constant M. This is a result of N. Kryloff-N. Bogolioùboff. (The existence of the mean sojourn.) Moreover, the integral operators T_{λ_i} $(i = 1, 2, \cdots, k)$ defined by the kernels $P_{\lambda_i}(t, E)$ are all strongly completely continuous.

§4.3. Structure of the kernel $P_1(t, E)$. By (4.5) and (4.14), we have

$$P_1(t, E) \geqq 0, \qquad P_1(t, \Omega) \equiv 1, \tag{4.16}$$

and the proper value equation in (**M**):

$$T(x) = x: \quad x(E) = \int_\Omega x(dt) P(t, E) \tag{4.17}$$

admits a non-trivial solution $x \neq 0$. In fact, by (4.10) of Theorem 5, for any $t_0 \in \Omega$, $P_1(t_0, E)$ is a solution of (4.17):

$$P_1(t_0, E) = \int_\Omega P_1(t_0, dt) P(t, E), \tag{4.18}$$

and $P_1(t, E)$ is not identically zero.

In this section, we shall study the general form of such a solution of (4.17), and using the results thus obtained, the structure of the kernel $P_1(t, E)$ will be determined.

In order to make our discussions clearer, we shall make use of some elementary notions from the theory of semi-ordered Banach spaces. A completely additive real-valued set function $x(E) \in$ (**M**) is called to be positive and is denoted by $x \geqq 0$, if we have $x(E) \geqq 0$ for any Borel set E of Ω; and for any pair of real-valued set functions $x(E), y(E) \in$ (**M**), denote by $x \geqq y$ the relation $x - y \geqq 0$.

Then the relation $x \geqq y$ determines a semi-ordering of the real Banach space $(\mathbf{M})$.

For any real-valued set function $x(E) \in (\mathbf{M})$, its total variation $\bar{x}(E)$ is a positive element of $(\mathbf{M})$, and if we put

$$x_+(E) = \tfrac{1}{2}(\bar{x}(E) + x(E)), \qquad x_-(E) = \tfrac{1}{2}(\bar{x}(E) - x(E)),$$

then these are also the positive elements of $(\mathbf{M})$. These are called the positive and the negative variations of x (on E) respectively. We have $x = x_+ - x_-$, $\bar{x} = x_+ + x_-$ and, from the trivial relation: $(-x)_+ = x_-$, we have $x - (x - y)_+ = y - (y - x)_+$. This last common element is called the minimum of two elements x and y, and is denoted by $x \wedge y$.[18] The special case $x \wedge y = 0$ requires our special attention. It is to be remarked that for two positive elements x and y of $(\mathbf{M})$, $x \wedge y = 0$ is equivalent to the condition that there exist two disjoint Borel sets E_1 and E_2 of Ω such that $x(E_1) = x(\Omega)$ and $y(E_2) = y(\Omega)$. This fact is needed in the following discussions. It is also to be noted that we have always $x_+ \wedge x_- = 0$, $(x - y)_+ \wedge (y - x)_+ = 0$ and

$$(x - (x \wedge y)) \wedge (y - (x \wedge y)) = 0. \tag{4.19}$$

Lemma 4.4. *If x and y are two real-valued solutions of* (4.17), *then $\bar{x}$, x_+, x_- and $x \wedge y$ are also solutions of* (4.17).

Proof. It is sufficient to prove this for $\bar{x}$. Since $P(t, E) \geqq 0$ for any Borel set $E \subset \Omega$, we have

$$\bar{x}(E) \leqq \int_\Omega \bar{x}(dt)P(t, E),$$

and, since $P(t, \Omega) \equiv 1$, here must stand the equal sign.

Lemma 4.5. *There exists a system of completely additive set functions $\{x_\alpha(E)\}$ $(\alpha = 1, 2, \cdots, l) \in (\mathbf{M})$, with the properties*:

$$T(x_\alpha) = x_\alpha, \quad x_\alpha \geqq 0, \quad x_\alpha(\Omega) = 1, \quad x_\alpha \wedge x_\beta = 0 \quad (\alpha \neq \beta), \tag{4.20}$$

such that any $x(E) \in (\mathbf{M})$ which satisfies

$$T(x) = x, \quad x \geqq 0, \quad x(\Omega) = 1, \tag{4.21}$$

is uniquely expressed as a linear combination:

$$x(E) = \sum_{\alpha=1}^{l} c_\alpha x_\alpha(E), \quad c_\alpha \geqq 0, \sum_{\alpha=1}^{l} c_\alpha = 1. \tag{4.22}$$

Proof. Let l be the maximum number of elements $x_1, x_2, \cdots, x_l \in (\mathbf{M})$ which satisfy (4.20). The existence of such an l is clear from the quasi-strong complete continuity of the operation T. For, such $x_1, x_2, \cdots, x_l$ are clearly mutually linearly independent and the proper space of T belonging to the proper value 1 is of finite dimension by Lemma 3.2 of §3.

[18] Indeed, it will be easily seen that $x \wedge y$ is the minimum of x and y in the sense of lattice, that is, $x \wedge y \leqq x$, $x \wedge y \leqq y$, and for any $z \in (M)$ with $z \leqq x$, $z \leqq y$, we have $z \leqq x \wedge y$.

We shall prove that this system $\{x_\alpha(E)\}$ $(\alpha = 1, 2, \cdots, l)$ is just the required one. In order to show this, let $x(E)$ be an arbitrary element of $(\mathbf{M})$ which satisfies (4.21). We shall show that $x(E)$ is a linear combination of $x_1, x_2, \cdots, x_l$. For this purpose, consider the minimum $x'_\alpha = x \wedge x_\alpha$ of x and x_α for $\alpha = 1, 2, \cdots, l$. By Lemma 4.4, x'_α is also a solution of (4.17). We shall first show that each x'_α is a constant multiple of $x_\alpha : x'_\alpha = c_\alpha x_\alpha$. Indeed, if this is not true for some α, then x_α and $x''_\alpha \equiv x'_\alpha/\| x'_\alpha \|$ are not equal, and consequently $(x_\alpha - x''_\alpha)_+$ and $(x''_\alpha - x_\alpha)_+$ are both $\neq 0$. Again by Lemma 4.4, these are also the solutions of (4.17). Hence, if we put $x_{\alpha 1} = (x_\alpha - x''_\alpha)_+/\| (x_\alpha - x''_\alpha)_+ \|$, $x_{\alpha 2} = (x''_\alpha - x_\alpha)_+/\| (x''_\alpha - x_\alpha)_+ \|$, then the system of $l + 1$ elements $x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha 1}, x_{\alpha 2}, x_{\alpha+1}, \cdots, x_l$ clearly satisfies (4.20), and this is a contradiction to the definition of l.

Thus we have proved that each $x'_\alpha = x \wedge x_\alpha$ is expressed in the form: $x'_\alpha = c_\alpha x_\alpha$, where c_α is a real number with $0 \leqq c_\alpha \leqq 1$. Next we shall prove that we have $x = \sum_{\alpha=1}^l x'_\alpha \equiv \sum_{\alpha=1}^l c_\alpha x_\alpha$. For this purpose, we shall show that $x' = x - \sum_{\alpha=1}^l x'_\alpha$ satisfies $x' \wedge x_\alpha = 0$ for $\alpha = 1, 2, \cdots, l$. In order to prove this, it is sufficient to show that we have $(x - x'_\alpha) \wedge x_\alpha = 0$ for $\alpha = 1, 2, \cdots, l$. This is, however, clear if $c_\alpha = 1$; for, $x'_\alpha \equiv x \wedge x_\alpha = x_\alpha$ implies $x = x_\alpha$ (since $x(\Omega) = x_\alpha(\Omega) = 1$), and $x' = x - x'_\alpha = 0$. And, in case $c_\alpha < 1$, this follows from $(1 - c_\alpha)((x - x'_\alpha) \wedge x_\alpha) \leqq (x - x'_\alpha) \wedge (1 - c_\alpha)x_\alpha = (x - x'_\alpha) \wedge (x_\alpha - x'_\alpha) = 0$ (by (4.19)). Thus $x' \wedge x_\alpha = 0$ is proved for $\alpha = 1, 2, \cdots, l$. Consequently, if we have $x' \neq 0$, then the system of $l + 1$ elements $x'/\| x' \|, x_1, x_2, \cdots, x_l$ will again satisfy (4.20), and this is also a contradiction.

Thus we have proved that we have $x' = 0$, and consequently $x = \sum_{\alpha=1}^l x'_\alpha = \sum_{\alpha=1}^l c_\alpha x_\alpha$. Since the uniqueness of the expression and the condition $\sum_{\alpha=1}^l c_\alpha = 1$ are both clear, the proof of Lemma 4.5 is hereby completed.

Corollary. *$\{x_\alpha(E)\}$ $(\alpha = 1, 2, \cdots, l)$ is a base of all the solutions of the proper value equation* (4.17); i.e., *any $x \in (\mathbf{M})$ which satisfies* (4.17) *is uniquely expressed as a linear combination of $x_1, x_2, \cdots, x_l$.*

Proof. Clear from Lemmas 4.4 and 4.5.

Theorem 6. *$P_1(t, E)$ is expressible in the form:*

$$P_1(t, E) = \sum_{\alpha=1}^{l} y_\alpha(t) x_\alpha(E), \tag{4.23}$$

where $\{x_\alpha(E)\}$ $(\alpha = 1, 2, \cdots, l)$ is the system of completely additive set functions ϵ $(\mathbf{M})$, which was defined in Lemma 4.5, and $\{y_\alpha(t)\}$ $(\alpha = 1, 2, \cdots, l)$ is a system of bounded Borel measurable functions ϵ (M^*), which satisfy

$$\overline{T}(y_\alpha) = y_\alpha, \qquad y_\alpha(t) \geqq 0, \qquad \sum_{\alpha=1}^{l} y_\alpha(t) = 1, \tag{4.24}$$

and

$$\int_\Omega x_\alpha(dt) y_\beta(t) = 1 \quad \text{or } 0 \text{ according as } \alpha = \beta \text{ or } \alpha \neq \beta. \tag{4.25}$$

Moreover, $\{y_\alpha(t)\}$ $(\alpha = 1, 2, \cdots, l)$ *is a base of all the solutions of the proper value equation in* (M^*):

$$\bar{T}(y) = y: y(t) = \int_\Omega P(t, ds)y(s), \tag{4.26}$$

and any solution of (4.26) *with* $y(t) \geqq 0$ *can be expressed uniquely in the form*:

$$y(t) = \sum_{\alpha=1}^{l} c_\alpha y_\alpha(t) \tag{4.27}$$

with non-negative constants c_α $(\alpha = 1, 2, \cdots, l)$.

PROOF. Since $P_1(t, E)$ is a solution of (4.17) for any $t \in \Omega$, the expression (4.23) follows directly from Lemma 4.5. If we take a Borel set E'_α such that $x_\alpha(E'_\alpha) = 1$ and $x_\beta(E'_\alpha) = 0$ for any $\beta \neq \alpha$ (the existence of such E'_α follows from the fact that $x_\alpha(\Omega) = 1$ and $x_\beta \wedge x_\alpha = 0$ for any $\beta \neq \alpha$), then (4.23) becomes $P_1(t, E'_\alpha) = y_\alpha(t)$. Hence each $y_\alpha(t)$ is a bounded Borel measurable function of t. We shall next prove (4.24). Since the second and the third relation of (4.24) are clear from (4.16), we have only to prove the first one. From (4.10) of Theorem 5 we have

$$\int_\Omega P(t, ds)P_1(s, E) = P_1(t, E),$$

or, by (4.23),

$$\sum_{\alpha=1}^{l}\left(\int_\Omega P(t, ds)y_\alpha(s)\right)x_\alpha(E) = \sum_{\alpha=1}^{l} y_\alpha(t)x_\alpha(E),$$

and, if we put $E = E'_\alpha$, then we have

$$\int_\Omega P(t, ds)y_\alpha(s) = y_\alpha(t).$$

Thus (4.24) is proved. In order to prove (4.25), start from the relation $T(x_\alpha) = x_\alpha$. Since $T(x) = x$ is equivalent to $T_1(x) = x$, we have $T_1(x_\alpha) = x_\alpha$, or

$$\int_\Omega x_\alpha(dt)P_1(t, E) = x_\alpha(E),$$

and, putting $E = E'_\beta$, we have the required relation (4.25).

Thus the first part of the theorem is proved. The second part may be proved as follows: Let $y(t)$ be a solution of (4.26). From (4.26) we have (exactly as in the preceding case)

$$y(t) = \int_\Omega P_1(t, ds)y(s),$$

and, by (4.23),

$$y(t) = \sum_{\alpha=1}^{l} y_\alpha(t)\left(\int_\Omega x_\alpha(ds)y(s)\right) = \sum_{\alpha=1}^{l} c_\alpha y_\alpha(t)$$

with $c_\alpha = \int_\Omega x_\alpha(ds)y(s)$. Since $y(s) \geqq 0$ implies $c_\alpha \geqq 0$, the proof of Theorem 6 is hereby completed.

§4.4. Ergodic decomposition of Ω under the condition (K). Let us consider the sets $\bar{E}_\alpha = E_t[y_\alpha(t) = 1]$, $\alpha = 1, 2, \cdots, l$. By (4.24), these are mutually disjoint Borel sets.

THEOREM 7. *There exists a system of mutually disjoint Borel sets* $\{\bar{E}_\alpha\}$ ($\alpha = 1, 2, \cdots, l$), *such that*

$$x_\alpha(\bar{E}_\beta) = 1 \text{ or } 0 \text{ according as } \alpha = \beta \text{ or } \alpha \neq \beta, \tag{4.28}$$

$$P(t, \bar{E}_\alpha) = 1, \qquad t \in \bar{E}_\alpha, \tag{4.29}$$

$$\underset{t \in \bar{E}_\alpha, E \subset \Omega}{\text{l.u.b.}} \left| \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t, E) - x_\alpha(E) \right| \leqq \frac{M}{n}, \qquad n = 1, 2, \ldots, \tag{4.30}$$

where M is a constant which is independent of n.

REMARK. (4.29) means that, for any α, each point $t \in \bar{E}_\alpha$ is transferred by the Markoff's process $P(t, E)$ inside $\bar{E}_\alpha$, and (4.30) means that the uniform limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t, E) = P_1(t, E)$$

is independent of the initial point $t \in \bar{E}_\alpha$. Because of these properties, $\bar{E}_\alpha$ ($\alpha = 1, 2, \cdots, l$) will be called the *ergodic parts* of Ω.

PROOF OF THEOREM 7. (4.28) and (4.29) are the consequences of (4.25) and (4.26) respectively. In order to show this, we have only to prove

LEMMA 4.6. *If $x(E)$ is a completely additive real-valued set function $\in$ (**M**) such that $x(\Omega) = 1$ and $x(E) \geqq 0$ for any Borel set $E \subset \Omega$, and if $y(t)$ is a bounded Borel measurable real-valued function $\in$ (M^*) such that $0 \leqq y(t) \leqq 1$ for any $t \in \Omega$, then*

$$\int_\Omega x(dt)y(t) = 1$$

implies

$$x(E_0) = 1, \qquad \textit{where } E_0 = \underset{t}{E}[y(t) = 1].$$

PROOF. If we put

$$E_n = \underset{t}{E}\left[1 - \frac{1}{n} \leqq y(t) < 1 - \frac{1}{n+1}\right],$$

then we have $\Omega = E_0 + \sum_{n=1}^{\infty} E_n$ and

$$1 = \int_{\Omega} x(dt)y(t) = \int_{E_0} x(dt)y(t) + \sum_{n=1}^{\infty} \int_{E_n} x(dt)y(t)$$
$$\leqq x(E_0) + \sum_{n=1}^{\infty} \left(1 - \frac{1}{n+1}\right) x(E_n)$$
$$\leqq x(E_0) + \sum_{n=1}^{\infty} x(E_n) = x(\Omega) = 1.$$

Since the equality holds only if we have $x(E_n) = 0$ for $n = 1, 2, \cdots$, we have $x(E_0) = 1$. Thus Lemma 4.6 and hereby (4.28) and (4.29) are proved.

(4.30) is a restatement of the relation (4.9) of Theorem 5 ($i = 1, \lambda_1 = 1$):

$$\underset{t \in \Omega, E \subset \Omega}{\text{l.u.b.}} \left| \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t, E) - P_1(t, E) \right| \leqq \frac{M}{n}, \qquad n = 1, 2, \cdots,$$

if we only observe that we have

$$P_1(t, E) = x_\alpha(E) \quad \text{for} \quad t \in \bar{E}_\alpha.$$

As is easily seen, $\bar{E}_\alpha$ is not necessarily the set of the smallest measure with the property (4.28). Indeed, there might exist a Borel set $E \subset \bar{E}_\alpha$ such that mes $(E) <$ mes $(\bar{E}_\alpha)$ and $x_\alpha(E) = 1$. If we denote by $E_\alpha(\subset \bar{E}_\alpha)$ the Borel set of the smallest measure among those which satisfy (4.28), then E_α is determined up to a set of measure zero and $E \subset E_\alpha$, mes $(E) > 0$ imply $x_\alpha(E) > 0$. We shall show that, if we take suitably the sets $E_\alpha \subset \bar{E}_\alpha$ ($\alpha = 1, 2, \cdots, l$), then the following theorem is true:

THEOREM 8. *The Borel sets E_α ($\alpha = 1, 2, \cdots, l$) and $\Delta = \Omega - \sum_{\alpha=1}^{l} E_\alpha$ satisfy*

$$P_1(t, E_\alpha) = 1, \qquad t \in \bar{E}_\alpha, \tag{4.31}$$

$$P(t, E_\alpha) = 1, \qquad t \in E_\alpha, \tag{4.32}$$

$$\begin{cases} \text{for any } t \in \bar{E}_\alpha \text{ and } E \subset E_\alpha, \text{ mes } (E) > 0 \text{ implies} \\ P_1(t, E) > 0 \text{ and consequently there exists a positive integer } n = n(t, E) \text{ such that } P^{(n)}(t, E) > 0, \end{cases} \tag{4.33}$$

$$\underset{t \in \Omega}{\text{l.u.b.}} \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t, \Delta) \leqq \frac{M}{n}, \qquad n = 1, 2, \cdots, \tag{4.34}$$

where M is a constant which is independent of n.

REMARK. (4.31) means that, for any α, each point $t \in \bar{E}_\alpha$ is transferred finally into E_α (in the sense of arithmetic mean), and (4.32) means that each point $t \in E_\alpha$ is transferred inside E_α. Moreover, by (4.33), E_α is indecomposable into two sets with the property (4.32). In Theorem 12, we shall obtain a more precise result than (4.34):

$$\underset{t \in \Omega}{\text{l.u.b.}} P^{(n)}(t, \Delta) \leqq \frac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots,$$

with positive constants M and ϵ.

Because of these properties, E_α ($\alpha = 1, 2, \cdots, l$) and Ω will be called the *ergodic kernels* and the *dissipative part* of Ω respectively. It is to be noted that the ergodic kernels are only determined up to a set of measure zero (even under the condition (4.32)), while the ergodic parts $\bar{E}_\alpha$ are strictly determined by $\bar{E}_\alpha = \underset{t}{E}[y(t) = 1]$.

PROOF OF THEOREM 8. Let $E_\alpha^0 (\subset \bar{E}_\alpha)$ be the Borel set of the smallest measure among those which satisfy (4.28). E_α^0 clearly satisfies (4.31) and (4.33) (since $P_1(t, E) = x_\alpha(E)$ for $t \in \bar{E}_\alpha$), but (4.32) is not always automatically satisfied for any $t \in E_\alpha^0$. In order to obtain the required Borel set E_α, we shall construct a sequence of Borel sets $E_\alpha^0 \supset E_\alpha^1 \supset E_\alpha^2 \cdots \supset E_\alpha^n \supset \cdots$ by mathematical induction. Let E_α^n be already defined and assume that we have $x_\alpha(E_\alpha^n) = 1$. Then we define E_α^{n+1} as the set of all $t \in E_\alpha^n$ which satisfies $P(t, E_\alpha^n) = 1$. Since

$$\int_{E_\alpha^n} x_\alpha(dt)P(t, E_\alpha^n) = \int_\Omega x_\alpha(dt)P(t, E_\alpha^n) = x_\alpha(E_\alpha^n) = 1,$$

we have (by Lemma 4.6) $x_\alpha(E_\alpha^{n+1}) = 1$. If we now consider the set $E_\alpha^\infty = \prod_{n=1}^\infty E_\alpha^n$, then $E_\alpha = E_\alpha^\infty$ is the required set. For, we have

$$x_\alpha(E_\alpha^\infty) = \lim_{n \to \infty} x_\alpha(E_\alpha^n) = 1$$

and

$$P(t, E_\alpha^\infty) = \lim_{n \to \infty} P(t, E_0^n) = 1 \quad \text{for } t \in E_\alpha^\infty.$$

Thus we have proved the existence of the Borel set $E_\alpha \subset \bar{E}_\alpha$ which satisfies (4.31), (4.32) and (4.33). Since (4.34) is clear from

$$P(t, \Delta) = \sum_{\alpha=1}^{l} y_\alpha(t)x_\alpha(\Delta) = 0, \qquad t \in \Omega,$$

the proof of Theorem 8 is completed.

§4.5. Proper values of modulus 1 of the operator T.

THEOREM 9. *Under the condition* (K), *each proper value of modulus* 1 *of the operation T is a root of unity.*

PROOF. Let λ, $|\lambda| = 1$, be a proper value of the bounded linear operation T, which maps (**M**) into itself. By Theorem 5, λ may also be considered as a proper value of the bounded linear operation $\bar{T}$, which maps (M^*) into itself. Indeed, (since $P_\lambda(t, E)$ is not identically zero) if we put $x(t) \equiv P_\lambda(t, E)$ for a suitable Borel set E, then $x(t)$ is a non-trivial solution of the proper value equation:

$$\bar{T}(x) = \lambda x: \lambda x(t) = \int_\Omega P(t, ds)x(s). \tag{4.35}$$

$x(t)$ is clearly a complex-valued bounded Borel measurable function $\epsilon\ (M^*)$. We shall prove that λ is a root of unity.

In order to prove this, we shall first show that the real-valued bounded measurable function $\bar{x}(t) \equiv |x(t)| \epsilon\ (M^*)$ attains its maximum. From (4.35), we have

$$\bar{x}(t) \leqq \int_\Omega P(t, ds)\bar{x}(s)$$

and consequently

$$\bar{x}(t) \leqq \int_\Omega P^{(n)}(t, ds)\bar{x}(s), \qquad n = 1, 2, \cdots.$$

Taking the arithmetic mean and considering its limit, we have

$$\bar{x}(t) \leqq \int_\Omega P_1(t, ds)\bar{x}(s)$$

or, by (4.23),

$$\bar{x}(t) \leqq \sum_{\alpha=1}^{l} y_\alpha(t)\left(\int_\Omega x_\alpha(ds)x(s)\right) \equiv \sum_{\alpha=1}^{l} \xi_\alpha y_\alpha(t),$$

where $\xi_\alpha = \int_\Omega x_\alpha(ds)x(s)$ is a real non-negative number. Let ξ be the maximum of ξ_α $(\alpha = 1, 2, \cdots, l)$. Since $\sum_{\alpha=1}^{l} y_\alpha(t) \equiv 1$ by (4.24), we have $\bar{x}(t) \leqq \xi$ for any $t \in \Omega$. We shall prove that this ξ is attained by $\bar{x}(t)$ at some point $t_0 \in \Omega$. Indeed, by the definition of ξ, there exists at least one integer α such that $\xi = \int_\Omega x_\alpha(ds)x(s)$, and since $x_\alpha(E) \geqq 0$, $x_\alpha(\Omega) = 1$ and $\bar{x}(s) \leqq \xi$ for any $s \in \Omega$, there must exist, by Lemma 4.6, a point $t_0 \in \Omega$ such that $\bar{x}(t_0) = \xi$ (or more precisely, the set $E_0 = \underset{s}{E}[\bar{x}(s) = \xi]$ satisfies $x_\alpha(E_0) = 1$).

We have thus proved that there exists a point $t_0 \in \Omega$, such that $\bar{x}(t_0) = |x(t_0)| = \max_{t \in \Omega} |x(t)| = \xi$. We shall next prove that the set

$$E(n) = \underset{t}{E}[x(t) = \lambda^n x(t_0)] \tag{4.36}$$

satisfies

$$P^{(n)}(t, E(n)) = 1 \tag{4.37}$$

for $n = 1, 2, \cdots$. From (4.35), we have

$$\lambda^n x(t_0) = \int_\Omega P^{(n)}(t_0, ds)x(s)$$

and, dividing by $\lambda^n x(t_0)$ and taking the real part,

$$1 = \int_\Omega P^{(n)}(t_0, ds)R\left(\frac{x(s)}{\lambda^n x(t_0)}\right), \qquad n = 1, 2, \cdots.$$

Since $P^{(n)}(t_0, E) \geqq 0$, $P^{(n)}(t_0, \Omega) \equiv 1$ and $R\left(\frac{x(s)}{\lambda^n x(t_0)}\right) \leqq 1$ for any $s \in \Omega$, we have (4.37) by Lemma 4.6. For, we have

$$\underset{s}{E}\left[R\left(\frac{x(s)}{\lambda^n x(t_0)}\right) = 1\right] = \underset{t}{E}\,[x(t) = \lambda^n x(t_0)].$$

Thus we have proved (4.37) for $n = 1, 2, \cdots$. From this follows the existence of two positive integers m and n, such that $E(m)\cdot E(n) \neq 0$. For, if we have $E(m)\cdot E(n) = 0$ for any couple of integers m, n $(m \neq n)$, then we have

$$P^{(m)}(t_0, E(l)) = 0 \quad \text{for} \quad m \neq l, \tag{4.38}$$

and consequently, by (4.37),

$$P_1(t_0, E(l)) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t_0, E(l)) = 0.$$

Since $P_1(t_0, E)$ is completely additive for Borel sets E, this implies

$$P_1(t_0, \sum_{l=1}^{\infty} E(l)) = 0.$$

This is, however, a contradiction, since we have, by (4.37) and (4.38),

$$P^{(m)}(t_0, \sum_{l=1}^{\infty} E(l)) = 1, \qquad m = 1, 2, \cdots,$$

and consequently

$$P_1(t_0, \sum_{l=1}^{\infty} E(l)) = \lim_{m\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^{(m)}(t_0, \sum_{l=1}^{\infty} E(l)) = 1.$$

Thus we have $E(m)\cdot E(n) \neq 0$ for a certain couple of integers m, n $(m \neq n)$. Consequently we have (by (4.36)) $\lambda^m x(t_0) = \lambda^n x(t_0) (\neq 0)$ or $\lambda^{m-n} = 1$, and hereby the proof of Theorem 9 is completed.

§4.6. Decomposition of each ergodic part (and ergodic kernel) into subergodic parts (and subergodic kernels).

THEOREM 10. *Under the condition* (K), *there exists a positive integer N such that $P^{(nN)}(t, E)$ is decomposed into the form:*

$$P^{(nN)}(t, E) = P_1^*(t, E) + S^{*(n)}(t, E), \qquad n = 1, 2, \cdots, \tag{4.39}$$

in such a way that we have

$$\begin{aligned} \int_\Omega P^{(N)}(t, ds)P_1^*(s, E) &= \int_\Omega P_1^*(t, ds)P^{(N)}(s, E) \\ &= \int_\Omega P_1^*(t, ds)P_1^*(s, E) = P_1^*(t, E), \end{aligned} \tag{4.40}$$

$$\int_\Omega P_1^*(t, ds)S^*(s, E) = \int_\Omega S^*(t, ds)P_1^*(s, E) = 0 \tag{4.41}$$

and

$$\underset{t\in\Omega, E\subset\Omega}{\text{l.u.b.}} \mid S^{*(n)}(t, E) \mid \leqq \frac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots, \tag{4.42}$$

where M and ϵ are positive constants which are independent of n.

PROOF. In Theorem 9, we have seen that each proper value λ of modulus 1 of T is a root of unity. Since T has only a finite number of such proper values, there exists a (sufficient large) positive integer N such that $\lambda^N = 1$ for any proper value λ of modulus 1 of T. Hence the bounded linear operation T^N corresponding to the kernel $P^{(N)}(t, E)$ has no proper values of modulus 1 other than 1, and Theorem 10 is a direct consequence of Theorem 5.

REMARK. It is to be noted that we have, by (4.39) and (4.42),

$$\underset{t\in\Omega, E\subset\Omega}{\text{l.u.b.}} \mid P^{(nN)}(t, E) - P_1^*(t, E) \mid \leqq \frac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots \tag{4.43}$$

with positive constants M and ϵ.

Moreover, just as in Lemma 4.5, there exists a system of real-valued completely additive set functions $\{x_i^*(E)\}$ $(i = 1, 2, \cdots, L)$ ϵ $(\mathbf{M})$ with the properties:

$$T^N(x_i^*) = x_i^*, \quad x_i^* \geqq 0, \quad x_i^*(\Omega) = 1, \quad x_i^* \wedge x_j^* = 0 \qquad (i \neq j), \tag{4.44}$$

such that each $x^*(E)$ ϵ $(\mathbf{M})$ which satisfies

$$T^N(x^*) = x^*, \qquad x^* \geqq 0, \qquad x^*(\Omega) = 1 \tag{4.45}$$

is uniquely expressed in the form:

$$x^*(E) = \sum_{i=1}^L c_i^* x_i^*(E), \qquad c_i^* \geqq 0, \qquad \sum_{i=1}^L c_i^* = 1. \tag{4.46}$$

In particular, just as in Theorem 6, $P_1^*(t, E)$ is expressed in the form:

$$P_1^*(t, E) = \sum_{i=1}^L y_i^*(t) x_i^*(E), \tag{4.47}$$

where $\{y_i^*(t)\}$ $(i = 1, 2, \cdots, L)$ is a system of real-valued bounded Borel measurable functions ϵ (M^*) which satisfy

$$\overline{T}^N(y_i^*) = y_i^*, \qquad y_i^*(t) \geqq 0, \qquad \sum_{i=1}^L y_i^*(t) \equiv 1. \tag{4.48}$$

Let us denote by $\bar{E}_i^*$ the set $\underset{t}{E}[y^*(t) = 1]$, $i = 1, 2, \cdots, L$. These will be called the *subergodic parts* of Ω. Exactly as in Theorem 7, these are mutually disjoint Borel sets, and we have

$$(4.49)\qquad \begin{cases} x_i^*(\bar{E}_j^*) = 1 \text{ or } 0 \text{ according as } i = j \text{ or } i \neq j, \\ P^{(N)}(t, \bar{E}_i^*) = 1, \qquad t \in \bar{E}_i^*, \\ \underset{t \in \bar{E}_i^*, E \subset \Omega}{\text{l.u.b.}} \mid P^{(nN)}(t, E) - x_i^*(E) \mid \leqq \dfrac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots \end{cases}$$

with positive constants M and ϵ. More precisely we can prove

THEOREM 11. *The totality of all the subergodic parts* $\bar{E}_i^*$ $(i = 1, 2, \cdots, L)$ *is divided into* l *classes* $(\bar{E}_{\alpha_1}^*, \bar{E}_{\alpha_2}^*, \cdots, \bar{E}_{\alpha_{d_\alpha}}^*)$ $(\alpha = 1, 2, \cdots, l)$, *where* d_α *is a divisor of* N *with* $\sum_{\alpha=1}^{l} d_\alpha = L$, *in such a way that*

$$(4.50)\qquad P(t, \bar{E}_{\alpha_{i+1}}^*) = 1, \qquad t \in \bar{E}_{\alpha_i}^*, \qquad i = 1, 2, \cdots, d_\alpha(\alpha_{d_\alpha+1} = \alpha_1),$$

$$(4.51)\qquad \underset{t \in \bar{E}_{\alpha_i}^*, E \subset \Omega}{\text{l.u.b.}} \mid P^{(nd_\alpha)}(t, E) - x_{\alpha_i}^*(E) \mid \leqq \frac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots,$$

where M *and* ϵ *are positive constants which are independent of* n.

Moreover, all $\bar{E}_{\alpha_i}^*$ $(i = 1, 2, \cdots, d_\alpha)$ *belonging to the same class are contained in the same ergodic part* $\bar{E}_\alpha$, *and if we denote by* $x_{\alpha_i}^*(E)$ *and* $y_{\alpha_i}^*(t)$ *the corresponding elements of* $(\mathbf{M})$ *and* (M^*) *respectively (which are obtained by the arguments given above), then we have*

$$(4.52)\qquad T(x_{\alpha_i}^*) = x_{\alpha_{i+1}}^*, \qquad i = 1, 2, \cdots, d_\alpha(\alpha_{d_\alpha+1} = \alpha_1),$$

$$(4.53)\qquad \bar{T}(y_{\alpha_{i+1}}^*) = y_{\alpha_i}^*, \qquad i = 1, 2, \cdots, d_\alpha(\alpha_{d_\alpha+1} = \alpha_1),$$

$$(4.54)\qquad x_\alpha(E) = \frac{1}{d_\alpha} \sum_{i=1}^{d_\alpha} x_{\alpha_i}^*(E),$$

$$(4.55)\qquad y_\alpha(t) = \sum_{i=1}^{d_\alpha} y_{\alpha_i}^*(t).$$

REMARK 1. (4.50) means that, for any α, each point $t \in \sum_{i=1}^{d_\alpha} \bar{E}_{\alpha_i}^*$ is transferred, by the Markoff's process $P(t, E)$, cyclically in $\bar{E}_{\alpha_1}^*, \bar{E}_{\alpha_2}^*, \cdots, \bar{E}_{\alpha_{d_\alpha}}^*$; and (4.51) means that, in each $\bar{E}_{\alpha_i}^*$, $P^{(d_\alpha)}(t, E)$ defines a Markoff's process, whose n^{th} iterate $P^{(nd_\alpha)}(t, E)$ is uniformly convergent to a limit which is independent of the initial point $t \in \bar{E}_{\alpha_i}^*$.

REMARK 2. The equality $\bar{E}_\alpha = \sum_{i=1}^{d_\alpha} \bar{E}_{\alpha_i}^*$ is not necessarily true. Indeed, $D_\alpha = \bar{E}_\alpha - \sum_{i=1}^{d_\alpha} \bar{E}_{\alpha_i}^*$ is the set of all $t \in \Omega$ such that $y_\alpha(t) \equiv \sum_{i=1}^{d_\alpha} y_{\alpha_i}^*(t) = 1$ and $y_{\alpha_i}^*(t) < 1$ for $i = 1, 2, \cdots, d_\alpha$. From the proof of (4.51), we see that

$$(4.51')\qquad \underset{t \in E_\alpha, E \subset \Omega}{\text{l.u.b.}} \mid P^{(nd_\alpha)}(t, E) - \sum_{i=1}^{d_\alpha} y_{\alpha_i}^*(t) x_{\alpha_i}^*(E) \mid \leqq \frac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots$$

with positive constants M and ϵ.

PROOF OF THEOREM 11. We begin with some preliminary considerations. Since $T^N T(x_i^*) = T T^N(x_i^*) = T(x_i^*)$, $T(x_i^*)$ satisfies (4.43) for $i = 1, 2, \cdots, L$. Hence there exists a system of real constants c_{ij}, $i, j = 1, 2, \cdots, L$, such that

$$T(x_i^*) = \sum_{j=1}^{L} c_{ij} x_j^*, \qquad c_{ij} \geqq 0, \qquad \sum_{j=1}^{L} c_{ij} = 1. \tag{4.56}$$

Thus T may be considered as a linear transformation on the system $\{x_i^*(E)\}$ $(i = 1, 2, \cdots, L)$. Consider the matrix $C = (c_{ij})$, $i, j = 1, 2, \cdots, L$. This matrix clearly satisfies

$$C^N = \text{unit matrix}. \tag{4.57}$$

We shall show that C is a matrix composed of only 0 and 1, which defines a permutation of the indices $1, 2, \cdots, L$. Indeed, denoting by $c_{ij}^{(N-1)}$ the ij-elements of the matrix $C^{(N-1)}$, we have $\sum_{k=1}^{L} c_{ik}^{(N-1)} = 1$, $\sum_{k=1}^{L} c_{ik}^{(N-1)} c_{ki} = 1$ for $i = 1, 2, \cdots, L$. Since $0 \leqq c_{ki} \leqq 1$ for any k and i, there must exist, for each i, an index k_i such that $c_{k_i i} = 1$. In other words, each column of the matrix C must contain 1 at least once. Since $\sum_{i=1}^{L} c_{ki} = 1$ for each k, we must have $k_i \neq k_j$ for $i \neq j$. Hence, $(k_1, k_2, \cdots, k_L)$ is a permutation of the indices $(1, 2, \cdots, L)$, and consequently we have $c_{ki} = 0$ for $k \neq k_i$.

Thus we have proved that C defines a permutation of the indices $1, 2, \cdots, L$. Hence the indices $1, 2, \cdots, L$ are divided into $l'(\leqq L)$ classes, and each class is permuted cyclically inside itself by the matrix C. Let these classes be K_α $(\alpha = 1, 2, \cdots, l')$ and the number of the indices belonging to K_α be d_α. By (4.57), each d_α is a divisor of N.

For each α, consider the set of all the indices which belong to K_α. By a suitable numbering $\alpha_1, \alpha_2, \cdots, \alpha_{d_\alpha}$ of these indices, we must have $c_{\alpha_i \alpha_{i+1}} = 1$ and $c_{\alpha_i \alpha_j} = 0$ for $j \neq i + 1$ $(i = 1, 2, \cdots, d_\alpha$; once for all, we put $\alpha_{d_\alpha+1} = \alpha_1)$. Consequently, we have (4.52).

After these preliminaries, we shall proceed to the proof of Theorem 11. We shall first prove that we have $l = l'$ and that there is a one-to-one correspondence between the ergodic parts $\bar{E}_\alpha$ and the classes K_α in such a way that (4.52), (4.53), (4.54) and (4.55) are true. ((4.52) is already proved.)

For this purpose, recall that each $x_\alpha(E)$ satisfies $T(x_\alpha) = x_\alpha$. Hence $x_\alpha(E)$ satisfies (4.45), and consequently $x_\alpha(E)$ is uniquely expressed in the form:

$$x_\alpha(E) = \sum_{i=1}^{L} c_i^* x_i^*(E) \equiv \sum_{\alpha=1}^{l'} \sum_{i=1}^{d_\alpha} c_{\alpha_i}^* x_{\alpha_i}^*(E). \tag{4.58}$$

We shall first prove that there exists a class K_α such that $c_i^* = 1/d_\alpha$ for $i \in K_\alpha$ and $c_i^* = 0$ for $i \bar{\in} K_\alpha$. In the first place, it is clear that c_i^* is independent of i in each class K_α. For, since $T(x_\alpha) = x_\alpha$ and $T(x_{\alpha_i}) = x_{\alpha_{i+1}}$, we have, from (4.58),

$$x_\alpha(E) = \sum_{\alpha=1}^{l} \sum_{i=1}^{d_\alpha} c_{\alpha_i}^* x_{\alpha_{i+1}}^*(E),$$

and this implies $c_{\alpha_i}^* = c_{\alpha_{i+1}}^*$ for $i = 1, 2, \cdots, d_\alpha$. In the second place, all indices i with $c_i^* > 0$ belong to the same class K_α. For, if this is not the case, then $x_\alpha(E)$ will be decomposed into two non-trivial parts x_α' and x_α'', which are

both invariant under T: $x_\alpha = x'_\alpha + x''_\alpha$, $x'_\alpha > 0$, $x''_\alpha > 0$, $x'_\alpha \wedge x''_\alpha = 0$, $T(x'_\alpha) = x'_\alpha$, $T(x''_\alpha) = x''_\alpha$. If we put $x_{\alpha 1} = x'_\alpha/\| x'_\alpha \|$ and $x_{\alpha 2} = x''_\alpha/\| x''_\alpha \|$, then the system of $l + 1$ elements $x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha 1}, x_{\alpha 2}, x_{\alpha+1}, \cdots, x_l$ will satisfy the condition (4.20), and this is a contradiction to the definition of l. Thus there must exist a class K_α such that $c_i^* = c > 0$ for $i \in K_\alpha$ and $c_i^* = 0$ for $i \bar{\in} K_\alpha$. Since it is clear that we have $c = 1/d_\alpha$, the relation (4.54) is hereby proved.

Thus to each $x_\alpha(E)$ ($\alpha = 1, 2, \cdots, l$) there corresponds a class of indices K_α. Conversely, as is easily seen, to each class K_α there corresponds a completely additive set function $x_\alpha(E) = \dfrac{1}{d_\alpha} \sum_{i=1}^{d_\alpha} x^*_{\alpha_i}(E)$ (which clearly belongs to the system determined in Lemma 4.5) in such a way that the correspondence $x_\alpha(E) \leftrightarrow K_\alpha$ is one-to-one. Hence we must have $l = l'$, and the one-to-one correspondence between the ergodic part $\bar{E}_\alpha$ and the class of indices K_α is also established.

We shall next prove (4.55). From the relation

$$\frac{1}{N} \sum_{m=1}^{N} \int_\Omega P_1^*(t, ds) P^{(m)}(s, E) = P_1(t, E)$$

we have, by (4.23) and (4.47),

$$\frac{1}{N} \sum_{m=1}^{N} \sum_{\alpha=1}^{l} \sum_{i=1}^{d_\alpha} y^*_{\alpha_i}(t) \left(\int_\Omega x^*_{\alpha_i}(ds) P^{(m)}(s, E) \right) = \sum_{\alpha=1}^{l} y_\alpha(t) x_\alpha(E),$$

or, by (4.52),

$$\frac{1}{N} \sum_{m=1}^{N} \sum_{\alpha=1}^{l} \sum_{i=1}^{d_\alpha} y^*_{\alpha_i}(t) x^*_{\alpha_{i+m}}(E) = \sum_{\alpha=1}^{l} y_\alpha(t) x_\alpha(E).$$

Using the fact that d_α is a divisor of N, we have

$$\sum_{\alpha=1}^{l} \left(\sum_{i=1}^{d_\alpha} y^*_{\alpha_i}(t) \right) \left(\frac{1}{d_\alpha} \sum_{i=1}^{d_\alpha} x^*_{\alpha_i}(E) \right) = \sum_{\alpha=1}^{l} y_\alpha(t) x_\alpha(E),$$

or, by (4.54),

$$\sum_{\alpha=1}^{l} \left(\sum_{i=1}^{d_\alpha} y^*_{\alpha_i}(t) \right) x_\alpha(E) = \sum_{\alpha=1}^{l} y_\alpha(t) x_\alpha(E);$$

and this implies (4.55) (put $E = \bar{E}_\alpha$).

In order to prove (4.53), we start from the trivial relation:

$$\int_\Omega P(t, ds) P_1^*(s, E) = \int_\Omega P_1^*(t, ds) P(s, E).$$

By (4.45) and (4.52), this becomes

$$\sum_{\alpha=1}^{l} \sum_{i=1}^{d_\alpha} \left(\int_\Omega P(t, ds) y^*_{\alpha_i}(s) \right) x^*_{\alpha_i}(E) = \sum_{\alpha=1}^{l} \sum_{i=1}^{d_\alpha} y^*_{\alpha_i}(t) \left(\int_\Omega x^*_{\alpha_i}(ds) P(s, E) \right)$$

$$= \sum_{\alpha=1}^{l} \sum_{i=1}^{d_\alpha} y^*_{\alpha_i}(t) x^*_{\alpha_{i+1}}(E),$$

and, putting $E = \bar{E}^*_{\alpha_{i+1}}$, we have

$$\int_\Omega P(t, ds) y^*_{\alpha_{i+1}}(s) = y^*_{\alpha_i}(t);$$

i.e., (4.53) is proved.

Thus we have proved (4.52), (4.53), (4.54) and (4.55). From (4.55), it is clear that each $\bar{E}^*_{\alpha_i}$ is contained in $\bar{E}_\alpha$, and (4.50) is a direct consequence of (4.53) (use Lemma 4.6).

Lastly, we shall prove (5.51). For any positive integers n and k we have, by (4.6) and (4.37),

$$P^{(nN+kd_\alpha)}(t, E) = \int_\Omega P^{(nN)}(t, ds) P^{(kd_\alpha)}(s, E) = \int_\Omega P_1^*(t, ds) P^{(kd_\alpha)}(s, E)$$
$$+ \int_\Omega S^{*(n)}(t, ds) P^{(kd_\alpha)}(s, E).$$

Since we have

$$\int_\Omega P_1^*(t, ds) P^{(kd_\alpha)}(s, E) = \int_\Omega x^*_{\alpha_i}(ds) P^{(kd_\alpha)}(s, E) = x^*_{\alpha_i}(E)$$

for any $t \in \bar{E}^*_{\alpha_i}$, and since, by (4.40),

$$\left| \int_\Omega S^{*(n)}(t, ds) P^{(kd_\alpha)}(s, E) \right| \leqq \underset{t \in \Omega, E \subset \Omega}{\text{l.u.b.}} \ | S^{*(n)}(t, E) | \leqq \frac{M}{(1+\epsilon)^n}$$

for $n = 1, 2, \cdots$, we have

$$| P^{(nN+kd_\alpha)}(t, E) - x^*_{\alpha_i}(E) | \leqq \frac{M}{(1+\epsilon)^n}$$

for $t \in \bar{E}^*_{\alpha_i}$, $k = 1, 2, \cdots, N/d_\alpha$; $n = 1, 2, \cdots$, where M and ϵ are positive constants which are independent of n and k. Hence we have (4.51), by a suitable change of M and ϵ.

Thus Theorem 11 is completely proved.

REMARK 3. We can also define *subergodic kernel* $E^*_{\alpha_i}$ in each subergodic part $\bar{E}^*_{\alpha_i}$; namely, $E^*_{\alpha_i}$ is the set of the smallest measure which satisfies $x^*_{\alpha_i}(E^*_{\alpha_i}) = 1$. If we suitably take E_α and $E^*_{\alpha_i}$ (which are all determined only up to a set of measure zero), then we have

COROLLARY.

$$E_\alpha = \sum_{i=1}^{d_\alpha} E^*_{\alpha_i}, \qquad E^*_{\alpha_i} = E_\alpha \bar{E}^*_{\alpha_i}, \tag{4.59}$$

$$P(t, E^*_{\alpha_{i+1}}) = 1. \qquad t \in E^*_{\alpha_i}. \tag{4.60}$$

PROOF. (4.59) is clear from (4.54), and (4.60) follows from (4.29), (4.50) and the second relation of (4.59).

It is to be noted that these E_α and $E^*_{\alpha_i}$ are exactly the final sets (ensembles finals) and their cyclic subsets which were discussed by W. Doeblin [1].

Theorem 12.

$$\underset{t \in \Omega}{\text{l.u.b.}}\ P^{(n)}(t, \Delta) \leqq \frac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots, \tag{4.61}$$

where M and ϵ are positive constants which are independent of n.

Proof. Since each $x^*_{\alpha_i}(E)$ satisfies

$$x^*_{\alpha_i}(\Delta) \leqq \sum_{i=1}^{d_\alpha} x^*_{\alpha_i}(\Delta) = d_\alpha \cdot x_\alpha(\Delta) = 0,$$

we have

$$P^*_1(t, \Delta) = \sum_{i=1}^{L} y^*_i(t) x^*_i(\Delta) \equiv 0$$

for any $t \in \Omega$, and consequently we have, by Theorem 10,

$$\underset{t \in \Omega}{\text{l.u.b.}}\ P^{(nN)}(t, \Delta) \leqq \frac{M}{(1+\epsilon)^n}, \qquad n = 1, 2, \cdots.$$

Since $P^{(n)}(t, \Delta)$ is monotone decreasing in n, we can deduce from this easily the relation (4.61) (by a suitable change of M and ϵ).

§4.7. Deduction of the condition (K) from the condition (D).

Lemma 4.7. *Let us denote by $I(s_0)$ the closed interval $0 \leqq s \leqq s_0$. Then $Q(t, s) \equiv P(t, I(s))$ is Borel measurable as a function of two variables t and s in $0 \leqq t, s \leqq 1$.*

Proof. By assumption, $Q(t, s)$ is Borel measurable in t if s is fixed, and if t is fixed $Q(t, s)$ is monotone increasing in s and is continuous on the right: $\lim_{s \to s_0+0} Q(t, s) = Q(t, s_0)$.

We shall prove that, for any α, the set $E(\alpha) = \underset{(t,s)}{E}\,[Q(t, s) < \alpha]$ is Borel measurable as a two-dimensional point set. For this purpose, put $E_s(\alpha) = \underset{t}{E}\,[Q(t, s) < \alpha]$. Since $Q(t, s)$ is monotone increasing in s, we have $E_{s_1}(\alpha) \supset E_{s_2}(\alpha)$ for $s_1 < s_2$, and consequently

$$E(\alpha) = \sum_{s \in \Omega} E_s(\alpha) \times I(s),$$

where $\times$ denotes the Cartesian product. Since $Q(t, s)$ is continuous in s on the right, the section of $E(\alpha)$ by the straight line $t = t_0$ is, if not empty, a semi-open interval of the form: $0 \leqq s < s_0$. Hence we have

$$E(\alpha) = \sum_{\substack{s = \text{rational} \\ s \in \Omega}} E_s(\alpha) \times I(s),$$

and this shows that $E(\alpha)$ is Borel measurable as a two-dimensional point set.

LEMMA 4.8.[19] *Let $K(t, E)$ and $n(t, E)$ be two kernels with bounded density:*

$$K(t, E) = \int_E k(t, s)\,ds, \qquad |k(t, s)| \leqq \|K\|,$$

$$N(t, E) = \int_E n(t, s)\,ds, \qquad |n(t, s)| \leqq \|N\|,$$

where $k(t, s)$ and $n(t, s)$ are both bounded measurable functions defined on $0 \leqq t$, $s \leqq 1$. If we consider the corresponding integral operators K and N, which map the Banach space $(\mathbf{M})$ into itself, then the integral operator NK defined by the kernel $\int_E \left(\int_0^1 k(t, u)n(u, s)\,du\right) ds$ is strongly completely continuous as an operator which maps $(\mathbf{M})$ into itself.

PROOF. For any $x(E) \in (\mathbf{M})$, put $y = K(x)$ and $z = N(y) = NK(x)$. The set function $z(E)$ is absolutely continuous:

$$z(E) = \int_E z'(s)\,ds,$$

and its density $z'(s)$ is given by

$$z'(s) = \int_0^1 y(du)n(u, s) = \int_0^1 \left(\int_0^1 x(dt)k(t, u)\right) n(u, s)\,du.$$

Hence $\|x\| \leqq 1$ implies

$$\underset{0 \leqq s \leqq 1}{\text{l.u.b.}} |z'(s)| \leqq \|K\| \cdot \|N\|$$

and

$$\int_{-\infty}^{+\infty} |z'(s + \delta) - z'(s)|\,ds \leqq \|K\| \int_{-\infty}^{+\infty} \left(\int_0^1 |n(u, s + \delta) - n(u, s)|\,du\right) ds$$

if we put $n(u, s) = 0$ for $s < 0$ and $s > 1$.

Consequently, we have

$$\lim_{\delta \to 0} \int_{-\infty}^{+\infty} |z'(s + \delta) - z'(s)|\,ds = 0$$

uniformly for all $x(E) \in (\mathbf{M})$ with $\|x\| \leqq 1$. Hence, by a theorem of A. Kolmogoroff [1] and M. Riesz [1], (if we consider $z'(s)$ as an element of the Banach space (L)) the totality of all $z'(s)$ corresponding to the unit sphere $\|x\| \leqq 1$ of $(\mathbf{M})$ is strongly compact in (L). In other words, NK is strongly completely continuous as an operator which maps $(\mathbf{M})$ into (L). Since (L) is isometric to a closed linear subspace of $(\mathbf{M})$, NK is also strongly completely continuous as an operator which maps $(\mathbf{M})$ into itself. Thus the proof of Lemma 4.8 is completed.

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[19] K. Yosida, Y. Mimura and S. Kakutani [1].

LEMMA 4.9. *If $P(t, E)$ satisfies the condition* (D), *then we have*

$$P^{(d)}(t, E) = \int_E q(t, s)\, ds + R(t, E), \tag{4.62}$$

where $q(t, s)$ is a bounded $\left(\leqq \frac{1}{\eta}\right)$ Borel measurable function defined for $0 \leqq t$, $s \leqq 1$, and

$$0 \leqq R(t, E) \leqq 1 - b \tag{4.63}$$

for any $t \in \Omega$ and $E \subset \Omega$.

PROOF. Define $Q(t, s)$ newly by

$$Q(t, s) = P^{(d)}(t, I(s)).$$

By Lemma 4.7, $Q(t, s)$ is Borel measurable as a function of two variables t and s. Since $Q(t, s)$ is monotone in s for any t, $Q(t, s)$ is almost everywhere differentiable in s for any fixed t. If we put

$$p(t, s) = \lim_{n \to \infty} n\left(Q\left(t, s + \frac{1}{n}\right) - Q(t, s)\right)$$

and define $q(t, s)$ by

$$q(t, s) = \min\left(p(t, s), \frac{1}{\eta}\right),$$

then $q(t, s)$ is bounded $\left(\leqq \frac{1}{\eta}\right)$ and Borel measurable as a function of two variables t and s. We shall prove that the kernel $R(t, E)$ defined by

$$R(t, E) = P^{(d)}(t, E) - \int_E q(t, s)\, ds$$

satisfies (4.63) for any $t \in \Omega$ and $E \subset \Omega$.

For this purpose, we have only to notice that there exists for any $t \in \Omega$ a Borel set N_t of measure zero such that

$$P^{(d)}(t, E) = \int_E p(t, s)\, ds + P^{(d)}(t, N_t \cdot E)$$

for any Borel set $E \subset \Omega$. Then we have

$$\begin{aligned} 0 \leqq R(t, E) &= \int_E (p(t, s) - q(t, s))\, ds + P^{(d)}(t, N_t \cdot E) \\ &= \int_{E \cdot E_t(\eta)} p(t, s)\, ds + P^{(d)}(t, N_t \cdot E) \\ &\leqq P^{(d)}(t, E_t(\eta) + N_t), \end{aligned}$$

where $E_t(\eta) = \underset{s}{E}\left[p(t, s) > \frac{1}{\eta}\right]$. Since clearly mes $(E_t(\eta)) < \eta$ for any $t \in \Omega$, we have mes $(E_t(\eta) + N_t) < \eta$ and consequently, by (D), $0 \leqq R(t, E) \leqq 1 - b$ for any $t \in \Omega$ and $E \subset \Omega$.

THEOREM 13. *The condition* (D) *implies the condition* (K).

PROOF. By Lemma 4.9, we have

$$T^d = Q + R, \qquad \| R \| \leqq 1 - b, \tag{4.64}$$

where Q and R are the two integral operators which are defined by the kernels $Q(t, E) = \int_E q(t, s)\, ds$ and $R(t, E)$ respectively. If we expand $T^{dm} = (Q + R)^m$ in 2^m terms:

$$T^{dm} = Q^m + Q^{m-1}R + Q^{m-2}RQ + \cdots + QR^{m-1} + R^m,$$

then the terms which contain Q at least twice as factor are all strongly completely continuous. In order to see this, consider for example the term $RQRQ^{m-3}$. By (4.22), QR and Q^{m-3} are both integral operators with bounded density kernels. Hence, by Lemma 4.8, QRQ^{m-3} and consequently $RQRQ^{m-3}$ are strongly completely continuous. Since the number of terms which contain Q at most once as factor is $m + 1$, and since the norm of each such term is $\leqq (1 - b)^{m-1}$ by (4.64), we see that there exists, for each m, a strongly completely continuous operator V_m, which maps (**M**) into itself, such that

$$\| T^{dm} - V_m \| \leqq (m + 1)(1 - b)^{m-1}.$$

Since the right hand side converges to zero as $m \to \infty$, the proof of Theorem 13 is hereby completed.

REMARK. The converse of Theorem 13 is not true. To see this, take an arbitrary point $s_0 \in \Omega$, and define $P(t, E)$ by

$$\begin{aligned} P(t, E) &= 1 \quad \text{if} \quad s_0 \in E, \\ &= 0 \quad \text{if} \quad s_0 \,\bar{\in}\, E, \end{aligned}$$

for any $t \in \Omega$. This kernel $P(t, E)$ defines a strongly completely continuous integral operator, but the condition (D) is clearly not satisfied.

§4.8. Markoff's process with a finite number of possible states. Consider a Markoff's process with a finite number ($= N$) of possible states. Let p_{ij} $(i, j = 1, 2, \cdots, N)$ be the transition probability that the i^{th} state is transferred to the j^{th} state after the elapse of a unit-time. Then the transition probability $p_{ij}^{(n)}$ that the i^{th} state is transferred to the j^{th} state after the elapse of n unit-times is given recurrently by

$$p_{ij}^{(n)} = \sum_{k=1}^{N} p_{ik} p_{kj}^{(n-1)}, \qquad p_{ij}^{(1)} = p_{ij}, \tag{4.65}$$

and we have always

$$(4.66) \qquad p_{ij}^{(n)} \geqq 0, \qquad \sum_{j=1}^{N} p_{ij}^{(n)} = 1,$$

for $i, j = 1, 2, \cdots, N$; $n = 1, 2, \cdots$.

We shall investigate the asymptotic behavior of $p_{ij}^{(n)}$ for large n. This problem was discussed by many authors (see the introduction at the beginning of the paper), and sometimes direct methods were successful in this case. We shall, however, treat this problem as a special case of our general Markoff's process.

Consider the finitely valued function $p(t, s)$ defined in the square $0 \leqq t, s \leqq 1$ by

$$(4.67) \qquad p(t, s) = N \cdot p_{ij} \quad \text{for} \quad \frac{i-1}{N} \leqq t < \frac{i}{N}, \quad \frac{j-1}{N} \leqq s < \frac{j}{N},$$

$i, j = 1, 2, \cdots, N$ (in case $i = N$ or $j = N$, $<$ is to be replaced by $\leqq$). Then $P(t, E) = \int_E p(t, s)\, ds$ defines a simple Markoff's process on the interval $\Omega = (0, 1)$, and we have

$$(4.68) \qquad P^{(n)}(t, E) = \int_E p^{(n)}(t, s)\, ds,$$

where

$$(4.69) \qquad p^{(n)}(t, s) = N \cdot p_{ij}^{(n)} \quad \text{for} \quad \frac{i-1}{N} \leqq t < \frac{i}{N}, \quad \frac{j-1}{N} \leqq s < \frac{j}{N},$$

$i, j = 1, 2, \cdots, N$ (in case $i = N$ or $j = N$, $<$ is again to be replaced by $\leqq$).

Thus the Markoff's process $P = (p_{ij})$ $(i, j = 1, 2, \cdots, N)$ is reduced to the continuous case $P = P(t, E)$. Since it is clear that the corresponding integral operator T is strongly completely continuous in this case, we have, by the results obtained above,

THEOREM 14. (i) *The limit*

$$(4.70) \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} p_{ij}^{(m)} = p_{ij}^{(\infty)}$$

exists for any i and j, and there exists a constant M such that

$$(4.71) \qquad \left| \frac{1}{n} \sum_{m=1}^{n} p_{ij}^{(m)} - p_{ij}^{(\infty)} \right| \leqq \frac{M}{n}, \qquad i, j = 1, 2, \ldots, N; n = 1, 2, \cdots.$$

(ii) *There exists a system of mutually disjoint ergodic parts $\bar{E}_\alpha$ $(\alpha = 1, 2, \cdots, l)$ such that*

$$(4.72) \qquad \sum_{j \in \bar{E}_\alpha} p_{ij} = 1 \quad \text{if} \quad i \in \bar{E}_\alpha,$$

(4.73) $\qquad p_{ij}^{(\infty)}$ *is independent of* i *in each* $\bar{E}_\alpha$.

(iii) *Each ergodic part* $\bar{E}_\alpha$ *contains an ergodic kernel* E_α *such that*

$$\sum_{j \in E_\alpha} p_{ij}^{(\infty)} = 1 \quad \text{if} \quad i \in \bar{E}_\alpha, \tag{4.74}$$

$$\sum_{j \in E_\alpha} p_{ij} = 1 \quad \text{if} \quad i \in E_\alpha. \tag{4.75}$$

(iv) $\Delta = \Omega - \sum_{\alpha=1}^{l} E_\alpha$ *is called the dissipative part of* Ω *and we have*

$$p_{ij}^{(n)} \leqq \frac{M}{(1+\epsilon)^n} \qquad j \in \Delta,\, i = 1, 2, \cdots, N;\, n = 1, 2, \cdots \tag{4.76}$$

with positive constants M *and* ϵ.

(v) *To each ergodic part* $\bar{E}_\alpha$ *there corresponds a positive integer* d_α *such that*

$$| p_{ij}^{(nd_\alpha)} - d_\alpha p_{ij}^{(\infty)} | \leqq \frac{M}{(1+\epsilon)^n}, \qquad i \in \bar{E}_\alpha, j = 1, 2, \cdots, N; n = 1, 2, \cdots \tag{4.77}$$

with positive constants M *and* ϵ. *Moreover, each* $\bar{E}_\alpha$ *contains* d_α *(mutually disjoint) subergodic parts* $\bar{E}^*_{\alpha_1}, \bar{E}^*_{\alpha_2}, \cdots, \bar{E}^*_{\alpha_{d_\alpha}}$ *such that*

$$\sum_{j \in \bar{E}^*_{\alpha_{k+1}}} p_{ij} = 1 \quad \text{if} \quad i \in \bar{E}^*_{\alpha_k}, \qquad k = 1, 2, \cdots, d_\alpha \quad (\alpha_{d_\alpha+1} = \alpha_1). \tag{4.78}$$

(vi) *If we further put* $E^*_{\alpha_k} = E_\alpha \cdot \bar{E}^*_{\alpha_k}$ *for any* α *and* k, *then we have*

$$\sum_{j \in E^*_{\alpha_{k+1}}} p_{ij} = 1 \quad \text{if} \quad i \in E^*_{\alpha_k}, \qquad k = 1, 2, \cdots, d_\alpha \quad (\alpha_{d_\alpha+1} = \alpha_1). \tag{4.79}$$

Added in proof. Recently, N. Dunford and B. J. Pettis [1] obtained some new results concerning weakly completely continuous operators defined on the space (L). Among others, they proved that if K and N are weakly completely continous operators which map (L) into itself, then NK is strongly completely continuous. This result is more precise than Lemma 4.8.

MATHEMATICAL INSTITUTE
OSAKA IMPERIAL UNIVERSITY.

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ON TYPES OF "WEAK" CONVERGENCE IN LINEAR NORMED SPACES

BY INGO MADDAUS, JR.

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Introduction. In his paper "On a generalized notion of convergence in a Banach space," B. Vulich (Vulich 1, pp. 156–174) has pointed out that it is often possible to introduce a metric into a given concrete space with a given notion of convergence of a sequence of elements such that convergence in accordance with the metric is equivalent to the given convergence notion. However, it is not always possible to do this in terms of a metric which is a distance function (Vulich 1, p. 163). Consequently, Vulich has introduced a generalized metric by associating with each finite set or complex of a given linear set[1] a non-negative real number or norm. A set of six axioms imposed on this generalized metric and a limit notion defined in terms of it give rise to the concept of K-normed spaces.[2] The K-normed spaces are found to be special cases of Banach spaces (Banach, p. 53), and the K-convergence of a sequence of elements to a limiting element is found to imply Banach convergence to the same element. That is to say, K-convergence of a sequence to an element is stronger than Banach convergence of the sequence to the same element.

In this paper, also, the notion of the norm of a finite complex is employed. However, this norm is subjected to only three of the six axioms used in the definition of the K-normed spaces of Vulich. A limit notion for sequences is defined in terms of this norm. Among other things it is shown that point-wise convergence of a sequence of continuous functions to a continuous function on

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[1] By a linear set will be meant the usual one given in Banach's book on page 26.

[2] A linear set X is said to be K-normed if there is associated with each finite set or complex of its elements a non-negative real number or norm, written $\| (x_1, x_2, \cdots, x_n) \|$, which satisfies the following axioms:

Axiom A. If $x_i = x_j$, then $\| (x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) \| = \| (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_j, \cdots, x_n) \| = \| (x_1, \cdots, x_i, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n) \|$.

Axiom B. $\| (x) \| = 0$ implies $x = \theta$, where θ is the null element of the linear set.

Axiom C. $\| (x_1, \cdots, x_n, x_1', \cdots, x_n') \| \leqq \| (x_1, \cdots, x_n) \| + \| (x_1', \cdots, x_n') \|$.

Axiom D. $\| (x_1, \cdots, x_n) \| \leqq \| (x_1, \cdots, x_n, x_{n+1}) \|$.

Axiom E. $\| (x_1 + x_1', \cdots, x_n + x_n') \| \leqq \| (x_1, \cdots, x_n) \| + \| (x_1', \cdots, x_n') \|$.

Axiom F. $\| (ax_1, \cdots, ax_n) \| = | a | \cdot \| (x_1, \cdots, x_n) \|$.

By $x_n \rightarrow_k x$ it is meant that given any $e > 0$ there exists an $N(e)$ such that $n \geqq N(e)$ implies $\| (x_n - x, \cdots, x_{n+p} - x) \| < e$ for all $p \geqq 0$. The sequence $\{x_n\}$ is said to be K-convergent if given any $e > 0$ there exists an $N(e)$ such that if $n, m \geqq N(e)$ then $\| (x_n - x_m, \cdots, x_{n+p} - x_m) \| < e$ for all $p \geqq 0$.

A linear set is said to be a K-normed space if it is K-normed and if the limit notions are given by the definitions of the previous paragraph.

the finite closed interval reduces to the convergence of the sequence to the same function in terms of the norm of this paper. In all concrete cases considered it is seen that convergence of a sequence to an element in accordance with the limit notion to be defined is weaker than Banach convergence of the sequence to the element. Linear functionals and operations which are continuous in terms of this limit notion are studied.

1. Axioms and fundamental notions. Consider a linear set X of elements x such that there is associated with each finite subset or complex of elements of X a non-negative real number, called the norm of the complex and written $\|(x_1, x_2, \cdots, x_n)\|$ for the elements $x_1, x_2, \cdots, x_n$. This norm will satisfy the following axioms:

Axiom A. If $x_i = x_j$, then $\|(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n)\| = \|(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_j, \cdots, x_n)\| = \|(x_1, \cdots, x_i, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)\|$.

Axiom B. $\|(x)\| = 0$ implies $x = \theta$, where θ is the null element of the linear set X.

Axiom 3. If $a_1, a_2, \cdots, a_n$ are real constants, then $\|(a_1x_1, a_2x_2, \cdots, a_nx_n)\| \leqq \max_{1 \leqq i \leqq n} |a_i| \cdot \|(x_1, x_2, \cdots, x_n)\|$.

A linear set with the norm of a finite complex defined and satisfying Axioms A, B, and 3 will be said to be H-normed.

Definition 1. A sequence $\{x_n\}$ of elements of an H-normed set X will be said to be convergent in the H-sense, or to be H-convergent, if given any $e > 0$ and any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a $K_0 = K(e, \{x_{n_k}\})$ such that $k, s \geqq K_0$ imply

$$\|(x_{n_1} - x_{n_s}, \cdots, x_{n_k} - x_{n_s})\| < e. \tag{1}$$

Definition 2. A sequence $\{x_n\}$ of elements of an H-normed set X will be said to be convergent to the element $x \in X$ in the H-sense, or to have x as its H-limit, written $x_n \rightarrow_H x$ or $H\text{-}\lim_{n=\infty} x_n = x$, if given any $e > 0$ and any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a $K_0 = K(e, \{x_{n_k}\})$ such that $k \geqq K_0$ implies

$$\|(x_{n_1} - x, \cdots, x_{n_k} - x)\| < e. \tag{2}$$

An H-normed set for which the limit notions are given by Definitions 1 and 2 will be referred to as an H-normed space.

It is evident that in an H-normed space if a sequence converges or converges to an element x then every subsequence does likewise.

Remark 1.1. It should be noted that our Axioms A and B are the Axioms A and B of K-normed spaces and Axiom 3 is a consequence of the axioms of K-normed spaces (Vulich 2, p. 61). Axioms A, B, and 3 are independent (Vulich 2, p. 60), and the examples which will be given in §2–5 will illustrate that there are H-normed spaces which are not K-normed according to the same norm.

The set of all real numbers forms an H-normed space if $\| (x_1, x_2, \cdots, x_n) \| = \inf (|x_1|, |x_2|, \cdots, |x_n|)$, where by "inf" is meant the smallest of the numbers involved. If in this same set $x_n \to_B x$ means that given any $e > 0$ there exists an $N(e)$ such that $n \geqq N(e)$ implies that $|x_n - x| < e$, then we have

THEOREM 1.1. *In the set of all real numbers $x_n \to_H x$ is equivalent to $x_n \to_B x$.*

PROOF. For any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $\| (x_{n_1} - x, \cdots, x_{n_k} - x) \| = \inf (|x_{n_1} - x|, \cdots, |x_{n_k} - x|) \leqq |x_{n_k} - x|$. If $x_n \to_B x$ then this last expression approaches zero as $k \to \infty$. To prove the converse assume that $x_n \to_H x$ does not imply $x_n \to_B x$. Then there exists an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a positive number λ such that $|x_{n_k} - x| > \lambda > 0$ for $k = 1, 2, \cdots$. Then $0 < \lambda < \| (x_{n_1} - x, \cdots, x_{n_k} - x) \|$ for every integral k. This contradicts the assumption that $x_n \to_H x$.

REMARK 1.2. In general any linear set for which an F-metric (Banach, p. 35) is defined is an H-normed space if $\| (x_1, \cdots, x_n) \| = \inf [(x_1, \theta), \cdots, (x_n, \theta)]$. By Theorem 1.1 $x_n \to_H x$ is equivalent to $x_n \to x$ in accordance with the F-metric. Hence, any linear normed space[3] is also an H-normed space, and when the norm of a finite complex is defined as in this paragraph norm convergence of a sequence to an element is equivalent to H-convergence to the same element. Under the same circumstances norm convergence of a sequence is equivalent to H-convergence of the sequence.

The following theorems relate to H-normed spaces.

THEOREM 1.2. *If $x_n \to_H x$ and $x_n = x'_n$ for every n, then $x'_n \to_H x$.*

THEOREM 1.3. *If $x_n \to_H x$, then for any $x' \in X$, $x_n + x' \to_H x + x'$.*

REMARK 1.3. In Remark 3.1 an example will be given which will show that $x_n \to_H x$ and $y_n \to_H y$ do not imply that $x_n + y_n \to_H x + y$.

Because of Theorem 1.3 it is evident that the topology of an H-normed space is a uniform topology.

THEOREM 1.4. $\| (ax_1, \cdots, ax_n) \| = |a| \cdot \| (x_1, \cdots, x_n) \|$.

PROOF. $\| (ax_1, \cdots, ax_n) \| \leqq |a| \cdot \| (x_1, \cdots, x_n) \| = |a| \cdot \| (aa^{-1}x_1, \cdots, aa^{-1}x_n) \| \leqq |a| \cdot |a^{-1}| \cdot \| (ax_1, \cdots, ax_n) \| = \| (ax_1, \cdots, ax_n) \|$. This completes the proof.

THEOREM 1.5. *If $\{a_n\}$ is a sequence of real numbers such that $|a_n| \leqq M$, where M is independent of n, and if $x_n \to_H \theta$, then $a_n x_n \to_H \theta$.*

PROOF. Choose any infinite subsequence $\{a_{n_k} x_{n_k}\}$ of $\{a_n x_n\}$. Then $\| (a_{n_1} x_{n_1}, \cdots, a_{n_k} x_{n_k}) \| \leqq M \| (x_{n_1}, \cdots, x_{n_k}) \|$. Since $x_n \to_H \theta$ it follows that the right hand side of this inequality approaches zero as $k \to \infty$.

COROLLARY 1.51. *If $\{a_n\}$ is a sequence of real numbers such that $a_n \to_B 0$ and if $x_n \to_H \theta$, then $a_n x_n \to_H \theta$.*

COROLLARY 1.52. *If a is a real number and $x_n \to_H x$, then $ax_n \to_H ax$.*

[3] A linear normed space is a linear set X with the property that there is associated with each element x a non-negative real number or norm, designated by $\| x \|$, which is such that (1) $\| x \| = 0$ implies $x = \theta$, (2) $\| x + y \| \leqq \| x \| + \| y \|$, (3) $\| tx \| = |t| \cdot \| x \|$ for each real t. A sequence $\{x_n\}$ of X is said to be convergent to $x \in X$ in the norm or Banach sense if $\| x_n - x \| \to 0$ as $n \to \infty$.

2. The space (CH). Consider the set of all continuous functions defined on the closed interval $(0, 1)$, and let the norm of a finite complex of elements be given by $\| (x_1, \cdots, x_n) \| = \max_{0 \leqq t \leqq 1} \inf (| x_1(t) |, \cdots, | x_n(t) |)$. With this norm the set becomes an H-normed space and it will be designated by (CH).

THEOREM 2.1. *In the space* (CH) *a necessary and sufficient condition that* $x_n \to_H x$ *is that* $x_n(t) \to x(t)$ *for each* $t \in (0, 1)$.

PROOF. Because of Theorem 1.3 it suffices to prove this theorem for the case $x_n \to_H \theta$.

Necessity. Assume on the contrary that for some t_0, $\overline{\lim} \, | x_n(t_0) | = 2\lambda > 0$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with the property that $| x_{n_k}(t_0) | > \lambda$ for each integral k. Therefore, $\| (x_{n_1}, \cdots, x_{n_k}) \| \geqq \inf (| x_{n_1}(t_0) |, \cdots, | x_{n_k}(t_0) |) > \lambda > 0$ for each integral k. Consequently, θ is not an H-limit of $\{x_n\}$, in contradiction to the assumption of the theorem.

Sufficiency. Assume that there is a subsequence of $\{x_n\}$, which will also be designated by $\{x_n\}$, such that $\| (x_1, \cdots, x_n) \|$ is greater than a positive number λ for $n = 1, 2, \cdots$. Designate $\inf (| x_1(t) |, \cdots, | x_n(t) |)$ by $f_n(t)$. For a fixed value of n it is easily shown that $f_n(t)$ is a continuous function on the closed interval $(0, 1)$. Hence, there exists a t_n such that $f_n(t_n) = \max_{0 \leqq t \leqq 1} | f_n(t) | = \| (x_1, \cdots, x_n) \|$. By the Weierstrass–Bolzano Theorem it is possible to choose a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and a point t_0 such that $\lim_{k=\infty} t_{n_k} = t_0$.

Each $f_n(t_0) \geqq \lambda$. For if $f_{n'}(t_0) < \lambda$ there would exist an interval I about t_0 such that if $t \in I$ then $f_{n'}(t) < \lambda$. Now choose $n_k > n'$ and such that $t_{n_k} \in I$; then $f_{n_k}(t_{n_k}) < f_{n'}(t_{n_k}) < \lambda$, which is a contradiction. But $f_n(t) \leqq | x_n(t) |$ for all n and t, so $| x_n(t_0) | > f_n(t_0) \geqq \lambda$, in contradiction to the assumption that $x_n(t) \to 0$ for all $t \in (0, 1)$.

REMARK 2.1. If $x_1, x_2, \cdots, x_n$ and $x_1', x_2', \cdots, x_n'$ are two finite complexes of an H-normed space it does not follow that $\| (x_1 + x_1', x_2 + x_2', \cdots, x_n + x_n') \| \leqq \| (x_1, x_2, \cdots, x_n) \| + \| (x_1', x_2', \cdots, x_n') \|$. For consider the following complexes from the H-normed space (CH): $x = 2$, $x = 5$ and $x' = 5$, $x' = 2$. In this case the inequality sign is actually reversed. If the sign of x_1 is changed then the inequality sign is again reversed. Therefore, unlike the norm of the K- normed spaces of B. Vulich, the norm of a finite complex of a H-normed space does not satisfy a "triangle property," nor does it satisfy a "triangle property" with the sign reversed.

REMARK 2.2. In the space (CH) it can readily be shown by methods similar to those employed in the proof of Theorem 2.1 that a necessary and sufficient condition for the convergence of a sequence in the H-sense is: $| x_n(t) - x_m(t) | \to 0$ for each $t \in (0, 1)$. There are numerous familiar examples from the space (CH) which illustrate that H-convergence of a sequence does not imply that the sequence has an H-limit which is of the space. That is to say, an H-normed space is not necessarily complete with respect to the H-convergence notions.

REMARK 2.3. The norm of a finite number of elements of an H-normed space

is not necessarily a continuous function of its arguments; i.e., $x_n^{(i)} \to_H x^{(i)}$ for $i = 1, 2, \cdots, p$ does not imply $\lim_{n=\infty} \| (x_n^{(1)}, \cdots, x_n^{(p)}) \| = \| (x^{(1)}, \cdots, x^{(p)}) \|$. For consider the following sequence from (CH):

$$x_n(t) = x_n(0) = 0 \text{ for } 2/n \leqq t \leqq 1,$$

$$x_n(1/n) = n,$$

and $x_n(t)$ linear elsewhere $(n = 1, 2, \cdots)$ (Caratheodory, p. 171). This sequence is such that $x_n \to_H \theta$. Let a second sequence be given by $x_n'(t) = 1$ for each n and for $t \in (0, 1)$. Then $x_n' \to_H 1$. Moreover, $\| (x_n, x_n') \| = 1$ for each n, whereas $\| (\theta, 1) \| = 0$.

3. The space (L^pH), $p \geqq 1$. The norm of a finite complex which was used in the previous paragraph to make the set of all continuous functions on the finite closed interval an H-normed space was obtained by operating on the greatest lower bound of the absolute values of the elements involved with the norm which is generally employed to make the set of all continuous functions a Banach space (Banach, p. 11). To define $\| (x_1, \cdots, x_n) \|$ in an analogous manner for the set of all functions defined on the closed interval (0, 1) and whose pth powers are summable set it equal to $\left\{ \int_0^1 [\inf (| x_1(t) |, \cdots, | x_n(t) |)]^p \, dt \right\}^{1/p}$, where by "inf" is meant the greatest lower bound of the functions involved at every point of (0, 1).[4] With this interpretation of the norm of a finite complex the set of functions considered is an H-normed space. This space will be referred to as the space (L^pH), $p \geqq 1$. It will be seen from what follows that the H-convergence of a sequence of elements of (L^pH), $p \geqq 1$, to an element of the space is extremely weak.

LEMMA 3.1. *If $\{f_n\}$ is a sequence of summable functions defined on* (0, 1) *and such that*

(1) $f_n(t) \geqq 0$ *for each* n,

(2) $\{f_n(t)\}$ *is non-increasing,*

(3) $\lim_{n=\infty} \int_0^1 f_n(t) \, dt = 0$,

then $\lim_{n=\infty} f_n(t) = 0$ *except on a set of zero measure.*

THEOREM 3.1. *In (L^pH), $p \geqq 1$, $x_n \to_H x$ is equivalent to either of the following statements:*

(1) *For any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $\inf_k | x_{n_k}(t) - x(t) | = 0$ except on a set of measure zero which depends on the subsequence.*

[4] All the work of this section would go through without difficulty if the "inf" should be defined as the greatest lower bound of the functions involved except on a set of zero measure.

(2) *For any subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *there is a set* E *of zero measure such that for* $t_0 \notin E$ *there is a subsequence* $\{x_{n_{k_l}}\}$ *of* $\{x_{n_k}\}$ *such that* $\lim_{l \to \infty} x_{n_{k_l}}(t_0) = x(t_0)$.

PROOF. Because of Theorem 1.3 it is sufficient to prove this theorem for the case $x_n \to_H \theta$. We first show that $x_n \to_H \theta$ implies condition (2). Let $f_n^s(t) = [\inf(| x_s(t) |, \cdots, | x_n(t) |)]^p$ for each $t \in (0, 1)$. If s is constant then by Lemma 3.1 $\lim_{n=\infty} f_n^s(t) = 0$ except on a set E_s of zero measure. Let $E = \sum_{s=1}^{\infty} E_s$; then E is of zero measure and it is the set mentioned in the statement of the theorem. For consider any t_0 of the complement of E; $\lim_{n=\infty} f_n^s(t_0) = 0$ for each s, and in particular for $s_0 = 1$. Hence, there exists an integer $s_1 > s_0$ such that $| x_{s_1}(t_0) | < 1$. Consider $\{f_n^{s_1+1}\}$; $\lim_{n=\infty} f_n^{s_1+1}(t_0) = 0$, and so there exists an integer $s_2 > s_1$ such that $| x_{s_2}(t_0) | < \frac{1}{2}$. In this way an increasing sequence $\{s_i\}$ of integers can be built up in such a way that $| x_{s_i}(t_0) | < 1/i$. Then $\lim_{i=\infty} x_{s_i}(t_0) = 0$. Therefore, E is the set mentioned in statement (2). Since $x_{n_k} \to_H \theta$ is a consequence of $x_n \to_H \theta$, and since we may apply the above argument to $\{x_{n_k}\}$, our contention is proved.

It is obvious that statement (2) implies statement (1).

Now suppose that (1) holds when $x = \theta$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and let $f_k(t) = [\inf(| x_{n_1}(t) |, \cdots, | x_{n_k}(t) |)]^p$. Then (1) states that $\inf_k | x_{n_k}(t) | = \lim_{k=\infty} [f_k(t)]^{1/p} = 0$ except on a set E of zero measure. Hence, $\lim_{k=\infty} f_k(t) = 0$ except on E and $\lim_{k=\infty} \int_0^1 f_k(t)\, dt = 0$. Therefore, $x_n \to_H \theta$.

REMARK 3.1. The following example from (L^pH) shows that in an H-normed space the H-limit of a sequence is not necessarily unique. Let $\{x_n\}$ be defined by $x_n(t) = \frac{1}{2}(1 + r_n(t))$, where $r_n(t) = \operatorname{sgn}(\cos 2\pi nt)$.[5] If $E_n = E(x_n(t) = 1)$, then $\{E_n\}$ is a sequence of sets of closed intervals with no points common to any pair. This sequence has the property that the measure of the points common to each E_{n_k} of any subsequence $\{E_{n_k}\}$ is zero. Consequently, each $\{x_{n_k}\}$ of $\{x_n\}$ has the property that $\lim_{k=\infty} \int_0^1 [\inf(| x_{n_1}(t) |, \cdots, | x_{n_k}(t) |)]^p\, dt = 0$, and so $x_n \to_H \theta$. Now consider the sequence $\{x_n - 1\}$. By a study of the complements of the E_n sets—the sets on which $x_n - 1 = -1$—it is readily seen that $x_n - 1 \to_H \theta$. By Theorem 1.3 it follows that $x_n \to_H 1$. Hence the sequence $\{x_n\}$ has the H-limits θ and 1.

Consider the sequence $\{x_n\}$ just defined together with the sequence $\{x'_n\}$, where $x'_n = 1 - x_n$. Since $\{x'_n\}$ has the H-limits θ and 1 it follows that $H\text{-}\lim_{n=\infty} x_n + H\text{-}\lim_{n=\infty} x'_n$ may equal either θ or 1. But $H\text{-}\lim_{n=\infty} (x_n + x'_n) = 1$, so in an H-normed space $x_n \to_H x$ and $y_n \to_H y$ do not imply $(x_n + y_n) \to_H x + y$.

[5] The r_n functions are closely related to the well known Rademacher functions.

REMARK. 3.2. By giving different interpretations to $\|(x_1, \cdots, x_n)\|$ in the set of all functions defined on (0, 1) and whose pth powers are summable the H-limit notion will be found to have different meanings. For example, in the case of all summable functions defined on (0, 1) let $\|(x_1, \cdots, x_n)\| = \max_{0 \leqq t \leqq 1} \inf \left(\left| \int_0^t x_1(t)\,dt \right|, \cdots, \left| \int_0^t x_n(t)\,dt \right| \right)$. The set of all summable functions on (0, 1) is an H-normed space according to this norm and it will be designated by $(LH)_1$. Since $\int_0^t x(t)\,dt$ is a continuous function of t on (0, 1) it is evident from Theorem 2.1 that $x_n \rightarrow_H x$ is now equivalent to $\lim_{n=\infty} \int_0^t x_n(t)\,dt = \int_0^t x(t)\,dt$ for every $t \in (0, 1)$.

4. The space (cH). Consider the set of all convergent sequences $\{\xi_i\}$ of real numbers. Let an element x_n of the set be designated by $\{\xi_{ni}\}$ and let $\|(x_1, \cdots, x_n)\| = \sup_{1 \leqq i \leqq \infty} | a_i^n |$, where by "sup" is meant the least upper bound of the numbers involved, and $a_i^n = \inf (| \xi_{1i} |, \cdots, | \xi_{ni} |)$. Then the set is an H-normed space and it will be designated by (cH).

THEOREM 4.1. *A necessary and sufficient condition that* $x_n \rightarrow_H x$ in (cH) *is that*

(a) $$\lim_{n=\infty} \xi_{ni} = \xi_i \quad \text{for} \quad i = 1, 2, \cdots,$$

(b) $$\lim_{n=\infty} \lim_{i=\infty} \xi_{ni} = \lim_{i=\infty} \xi_i .$$

PROOF. This theorem can be derived from Theorem 2.1 since (cH) can be put into a 1-1 norm preserving correspondence with a subset of (CH). The correspondence is $\{\xi_i\}$ to $f(t)$ where $f(1/i) = \xi_i$, $f(0) = \lim_{i=\infty} \xi_i$ and $f(t)$ is linear elsewhere.[6]

5. The space (l^pH), $p \geqq 1$. Consider the set of all sequences $\{\xi_i\}$ of real numbers such that $\sum_{i=1}^{\infty} | \xi_i |^p < \infty$. Let any element x_n be designated by $\{\xi_{ni}\}$, and let $\|(x_1, \cdots, x_n)\| = [\sum_{i=1}^{\infty} | a_i^n |^p]^{1/p}$, where $a_i^n = \inf (| \xi_{1i} |, \cdots, | \xi_{ni} |)$. Then the set is an H-normed space and it will be designated by (l^pH), $p \geqq 1$.

THEOREM 5.1. *A necessary and sufficient condition that* $x_n \rightarrow_H x$ *in* (l^pH) *is that* $\lim_{n=\infty} \xi_{ni} = \xi_i$ *for* $i = 1, 2, \cdots$.

PROOF. This theorem can be derived from statement (1) of Theorem 3.1 since (l^pH), $p \geqq 1$, can be put into a 1-1 norm preserving correspondence with a subset of (L^pH), $p \geqq 1$. The correspondence is $\{\xi_i\}$ to $f(t)$, where $f(t) = i(i+1)\xi_i$ for $1/(1+i) < t < 1/i$ and $f(t) = 0$ elsewhere.

[6] The proofs of Theorems 4.1 and 5.1 were suggested by the referee. They are used here because they are much simpler than those originally given by the author.

6. Weak sequential convergence. In the last four sections certain concrete sets of elements have been considered as H-normed spaces by giving definite meanings to the norms of finite subsets of elements. Each of these sets may also be considered as a Banach space by proper definition of the norm for single elements. Banach has given necessary and sufficient conditions for weak sequential convergence[7] in each case considered (Banach, pp. 134–137). An examination of our Theorems 2.1, 3.1, 4.1, and 5.1 will readily show that the necessary and sufficient conditions for $x_n \rightarrow_H x$ are in each case weaker than the necessary and sufficient conditions for weak sequential convergence. That is to say, $x_n \rightarrow_H x$ for the sets considered has been made weaker than weak sequential convergence by the interpretations given the norms of finite complexes. In this section different interpretations of the norms of finite complexes on the same sets considered in the previous sections will be given, and the resulting H-convergence notions will be found to be equivalent to weak sequential convergence.

To this end it is advisable to prove first a general theorem on Banach spaces and then to give appropriate interpretations to the norms of finite complexes in the different cases to be considered. Let E be a Banach space and let $\bar{E}$ be its conjugate, or the space of all linear limited functionals defined on E (Banach, p. 54). If $\{f_n\}$ is a sequence of and f is a single linear limited functional defined on E, then $f_n \rightarrow f$ will always be taken to mean $f_n(x) \rightarrow f(x)$ for every $x \epsilon E$; i.e., $f_n \rightarrow f$ will mean that $\{f_n\}$ converges weakly to f (Banach, p. 122). Bounded sets of $\bar{E}$ will be weakly compact if every sequence chosen from a bounded (in the norm sense) set of linear limited functionals defined on E is such that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and another linear limited functional f_0 defined on E such that $f_{n_k} \rightarrow f_0$. It should be noted that if $f_n \rightarrow f$ and $\|f\| \leqq M$, where M is independent of n, then $\|f\| \leqq \underline{\lim} \|f_n\| \leqq M$ (Banach, p. 123).

In the Banach space E let $\|(x_1, \cdots, x_n)\| = \max\limits_{\|f\|=1} G_n(f)$, where f is a linear limited functional on E and where $G_n(f) = \inf (|f(x_1)|, \cdots, |f(x_n)|)$. $G_n(f)$ is a real valued function defined on $\bar{E}$. Because of the introduction of this norm for a finite complex the elements which form the Banach space E also form an H-normed space (EH). In this section the notions of H-convergence will be in terms of this norm.

LEMMA 6.1. *If $f_s \rightarrow f$ then $G_n(f_s) \rightarrow G_n(f)$.*

PROOF. This lemma follows readily from the fact that each $G_n(f)$ involves only a finite number of elements of E and from the definition of $f_s \rightarrow f$.

Let F be the subset of E such that if $f \epsilon F$ then $\|f\| = 1$. Let F_0 be the closed extension of F in the sense that if $f_n \epsilon F$ and $f_n \rightarrow f$ then $f \subset F_0$.

LEMMA 6.2. *If the Banach space E is such that in $\bar{E}$ bounded sets are weakly compact, then $\max\limits_{\|f\|=1} G_n(f)$ is attained by some element $f_0 \subset F_0$.*

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[7] The sequence $\{x_n\}$ of a space X is said to be convergent to an element $x \epsilon X$ in the weak sequential manner if for every linear (homogeneous and additive) and continuous functional $f(x)$ defined on X, $\lim\limits_{n=\infty} f(x_n) = f(x)$ (Banach, p. 133).

PROOF. Consider any $G_{n_0}(f)$ and a sequence $\{f_s\}$ of elements of F, and let $\lim_{s=\infty} G_{n_0}(f_s) = \max_{||f||=1} G_{n_0}(f)$. Each $G_{n_0}(f_s)$ is actually equal to the absolute value of f_s at some point of E; for by definition each $G_{n_0}(f_s)$ involves only a finite number of elements of E. Therefore, $G_{n_0}(f_s) = |f_s(x_i)|$, where i assumes one of the values $1, 2, \cdots, n_0$. Since the sequence $\{f_s\}$ is infinite there is some one x_{i_0} and an infinite subsequence $\{f_{s_p}\}$ of $\{f_s\}$ such that $G_{n_0}(f_{s_p}) = |f_{s_p}(x_{i_0})|$ for each integral p. Since each $||f_{s_p}|| = 1$ there is a subsequence $\{f_{s_{p_j}}\}$ of $\{f_{s_p}\}$ and an f_0, of unit norm or less, such that $f_{s_{p_j}} \to f_0$. By Lemma 6.1 it follows that $\lim_{j=\infty} G_{n_0}(f_{s_{p_j}}) = G_{n_0}(f_0)$. This completes the proof.

THEOREM 6.1. *If E is a Banach space such that in $\bar{E}$ bounded sets are weakly compact then $x_n \to_H x$ is equivalent to the weak sequential convergence of $\{x_n\}$ to x.*

PROOF. This theorem is a consequence of Lemmas 6.1 and 6.2 and its proof follows almost word by word the proof of Theorem 2.1.

It is well-known that the sets of elements which form the spaces (CH), (L^pH), (l^pH), $(p \geqq 1)$, and (cH) are also elements of separable Banach spaces if the norm for single elements is properly chosen. Banach states (Banach, p. 123) that each separable Banach space E has a conjugate space whose bounded sets are weakly compact as sets of functionals over E. Consequently, if the norm of a finite complex is defined as in this section for any one of the concrete Banach spaces just mentioned, then $x_n \to_H x$ is equivalent to the weak sequential convergence of $\{x_n\}$ to x.

7. Comparison of limit notions. It is of interest to note how the gap between the H-convergence in H-normed spaces and the K-convergence in K-normed spaces may be bridged. It is first necessary to introduce a new limit notion.

DEFINITION 1^0. A sequence $\{x_n\}$ of elements of an H-normed set X will be said to be convergent in the G-sense, or to be G-convergent, if given any $e > 0$ and any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a $K_0 = K(e, \{x_{n_k}\})$ such that for $k \geqq K_0$ there exists a $P_0 = P(e, k, \{x_{n_k}\})$ such that $k \geqq K_0$, $s \geqq K_0$ and $p \geqq P_0$ imply

$$(1^0) \quad ||(x_{n_k} - x_{n_s}, \cdots, x_{n_{k+p}} - x_{n_s})|| < e.$$

DEFINITION 2^0. A sequence $\{x_n\}$ of elements of an H-normed set X will be said to be convergent to the element $x \in X$ in the G-sense, or to have x as its G-limit, written $x_n \to_G x$ or $G\text{-}\lim_{n=\infty} x_n = x$, if given any $e > 0$ and any infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a $K_0 = K(e, \{x_{n_k}\})$ such that for $k \geqq K_0$ there exists a $P_0 = P(e, k, \{x_{n_k}\})$ such that $k \geqq K_0$ and $p \geqq P_0$ imply

$$(2^0) \quad ||(x_{n_k} - x, \cdots, x_{n_{k+p}} - x)|| < e.$$

THEOREM 7.1. *In a K-normed set $x_n \to_K x$ is equivalent to $x_n \to_G x$.*

PROOF. If $x_n \to_G x$ then given any $e > 0$ there exists an $N(e)$ and if $n \geqq N(e)$ there exists a $P(e, n)$ such that $n \geqq N(e)$ and $p \geqq P(e, n)$ imply

$\|(x_n - x, \cdots, x_{n+p} - x)\| < e$. But for all $0 \leqq i \leqq p$, $\|(x_n - x, \cdots, x_{n+i} - x)\| \leqq \|(x_n - x, \cdots, x_{n+p} - x)\| < e$, because of Axiom D of K-normed sets. Hence, the required inequality holds for every $i \geqq 0$, and so $x_n \to_K x$.

To prove the converse note that $x_n \to_K x$ implies $x_{n_i} \to_K x$ for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$. From this fact it is immediately evident that $x_n \to_K x$ implies $x_n \to_G x$.

THEOREM 7.2. *If in a K-normed set a sequence $\{x_n\}$ is K-convergent then it is G-convergent, and conversely.*

In the proofs of Theorems 2.1, 3.1, 4.1, 5.1, and 6.1 we made use of a property which when generalized to cover all cases may be written as

PROPERTY 1. $\|(x_1, \cdots, x_n, x_{n+1})\| \leqq \|(x_1, \cdots, x_n)\|$.

This property is actually Axiom D of K-normed sets with the inequality sign reversed. Since the norms of the spaces (CH), (L^pH), (l^pH), $(p \geqq 1)$, (cH) and (EH) satisfy Property 1 it is evident that they are not K-normed spaces.

THEOREM 7.3. *In an H-normed space whose norm satisfies Property* 1 $x_n \to_H x$ *is equivalent to* $x_n \to_G x$.

PROOF. The proof of this theorem depends on Property 1 in much the same manner that the proof of Theorem depends on Axiom D.

THEOREM 7.4. *In an H-normed space whose norm satisfies Property* 1 *if a sequence is G-convergent then it is H-convergent, and conversely.*

REMARK 7.1. Consequently, the G-convergence notions may be used in place of the H-convergence notions and the K-convergence notions in the mutually exclusive classes of H-normed sets satisfying Property 1 and of K-normed sets, respectively.

8. H^*-normed spaces. By an H^*-normed space will be meant an H- normed space whose elements also form a linear normed space and in which convergence of a sequence $\{x_n\}$ to x in the norm sense (Banach convergence), i.e., $\|x_n - x\| \to 0$—written $x_n \to_B x$—implies $x_n \to_H x$. If an H-normed space satisfies Property 1 and if it is linear normed with respect to the norm of a finite complex when that norm operates on single elements then it is evidently an H^*-normed space. The K-normed spaces are not necessarily H-normed spaces (Vulich 1, p. 163) for the K-limit notion is in general stronger than the limit notion in the norm sense.

REMARK 8.1. The example of Remark 3.2 is an H-normed space satisfying Property 1. Moreover, the set of elements of that example form a linear normed space when the norm $\|x\| = \max_{0 \leqq t \leqq 1} \left| \int_0^t x(u)\, du \right|$ is used, and so it is an H^*-normed space. However, this is not the usual norm employed to make the set of summable functions on (0, 1) a linear normed space—the usual norm is $\|x\| = \int_0^1 |x(u)|\, du$. Nevertheless, it is evident that the space $(LH)_1$ is an H^*-normed space when $x_n \to_B x$ is in accordance with the second norm just given and when $x_n \to_H x$ is with respect to the norm of a finite complex of $(LH)_1$.

By an $\bar{H}$-normed space will be meant an H^*-normed space in which the H-limit is unique and satisfies the condition: $x_n \to_H x$ and $y_n \to_H y$ imply $(x_n + y_n) \to_H (x + y)$.

REMARK 8.2. The space (L^pH), $p \geqq 1$, is an H^*-normed space which is not $\bar{H}$-normed.

LEMMA 8.1. *If $\{a_n\}$ is a sequence of real numbers such that $a_n \to_B a$ and if x is an element of an H-normed space whose norm satisfies Property* 1, *then $a_n x \to_H ax$.*

THEOREM 8.1. *If $\{x_n\}$ is a sequence of elements of an $\bar{H}$-normed space X whose norm satisfies Property* 1, *if $x_n \to_H x$ where $x \in X$, and if $\{a_n\}$ is a sequence of real numbers such that $a_n \to_B a$, then $a_n x_n \to_H ax$.*

PROOF. $a_n x_n - ax = (a_n x_n - a_n x) + (a_n x - ax)$. From Theorem 1.5 it follows that the first expression on the right approaches θ in the H-sense, and from Lemma 8.1 it follows that the second expression does likewise. The theorem follows from the fact that the space is $\bar{H}$-normed.

REMARK 8.3. An $\bar{H}$-normed space whose norm satisfies Property 1 does not have the property that the derived class is closed; that is to say, if $x_n^m \to_H x_n$ for $n = 1, 2, \cdots$, and if $x_n \to_H x$, it does not follow that there exists a subsequence $\{x_n^{m_n}\}$, where $m_1 < m_2 < \cdots < m_n < \cdots$, such that $x_n^{m_n} \to_H x$. For consider the following example from the space of all continuous functions defined on (0, 1) with the norm defined as in §6. It has been shown in §6 that in this space $x_n \to_H x$ is equivalent to the weak sequential convergence of $\{x_n\}$ to x; i.e., $x_n \to_H x$ is equivalent to $x_n(t) \to x(t)$ for each $t \in (0, 1)$ and to the existence of a constant $M > 0$ such that $\max_{0 \leqq t \leqq 1} | x_n(t) | \leqq M$ (Banach, p. 134). Let $\{x_n\}$ be defined as follows:

$$x_n(t) = x_n(0) = 0 \text{ for } 2/n \leqq t \leqq 1,$$

$$x_n(1/n) = 1,$$

$x_n(t)$ linear elsewhere. This sequence obviously has θ as its weak sequential limit. Let x_n^m be such that

$$x_n^m(t) = x_n(t) \text{ for } 0 \leqq t \leqq 2/n,$$

$$x_n^m(t) = 0 \text{ for } 2/n + 2/m \leqq t \leqq 1,$$

$$x_n^m(2/n + 1/m) = n,$$

$x_n^m(t)$ linear elsewhere. For each fixed value of n, $x_n^m \to x_n$ in the weak sequential sense. But since weak sequential convergence implies boundedness of the norms of elements it follows that it is impossible to find a sequence of integers $m_1 < m_2 < \cdots < m_n < \cdots$ such that $x_n^{m_n} \to_H \theta$.

9. Linear operations. Let X and Y be two sets of elements. If to each $x \in X$ there is made to correspond an element $y \in Y$ then an operation $y = U(x)$ is defined. X will be called the domain and Y the range of the operation.

DEFINITION 3. If $y = U(x)$ is an operation which transforms X into all or a part of Y, then if

(1) X and Y are H-normed spaces $y = U(x)$ is said to be HH-continuous at the point x_0 if $x_n \to_H x_0$ implies $U(x_n) \to_H U(x_0)$,
(2) X is an H-normed space and Y is a linear normed space $y = U(x)$ is said to be HB-continuous at the point x_0 if $x_n \to_H x_0$ implies $U(x_n) \to_B U(x_0)$,
(3) X is a linear normed space and Y is an H-normed space $y = U(x)$ is said to be BH-continuous at the point x_0 if $x_n \to_B x_0$ implies $U(x_n) \to_H U(x_0)$,
(4) X and Y are linear normed spaces $y = U(x)$ is said to be BB-continuous at the point x_0 if $x_n \to_B x_0$ implies $U(x_n) \to_B U(x_0)$.

If in any one of the above cases the specified continuity condition holds for every point of X then the operation is said to be continuous in that sense on X.

THEOREM 9.1. *If X is an H^*-normed space, Y is an H-normed space, and $y = U(x)$ is HH-continuous, then it is BH-continuous.*

PROOF. $x_n \to_B x$ implies $x_n \to_H x$ in an H^*-normed space, so by the HH-continuity of $y = U(x)$, $x_n \to_B x$ implies $U(x_n) \to_H U(x)$.

THEOREM 9.2. *If X is a linear normed space, Y is an H^*-normed space, and $y = U(x)$ is BB-continuous, then it is BH-continuous.*

THEOREM 9.3. *If X is an H-normed space, Y is an H^*-normed space, and $y = U(x)$ is HB-continuous then it is HH-continuous.*

THEOREM 9.4. *If the additive operation $y = U(x)$ transforms an H-normed space X into a space Y of the same type and is HH-continuous at a single point, then it is HH-continuous on X.*

THEOREM 9.5. *Statements similar to that of the previous theorem hold when $y = U(x)$ is HB-, BH-, or BB-continuous at a single point, X is H-normed, linear normed, or linear normed, respectively, and Y is linear normed, H-normed, or linear normed, respectively.*

THEOREM 9.6. *If $y = U(x)$ is an HH-continuous and additive operation from an $\bar{H}$-normed space X to an $\bar{H}$-normed space Y, then it is homogeneous.*

PROOF. The proof of this theorem depends on the uniqueness of the H-limit notion in $\bar{H}$-normed spaces.

THEOREM 9.7. *Statements similar to that of the previous theorem hold when $y = U(x)$ is additive and HB-, BH-, or BB-continuous, X is $\bar{H}$-normed, linear normed, or linear normed, respectively, and Y is linear normed, $\bar{H}$-normed, or linear normed, respectively.*

In case the operation is from a domain X to the set of all real numbers, which set is readily seen to be an $\bar{H}$-normed space when the norm of §1 is used, it will be called a functional and will be designated by $y = f(x)$. Since in the $\bar{H}$-normed space of real numbers (see §1) the H- and B-convergence notions are equivalent (see Theorem 1.1), the four types of operations of Definition 3 reduce to two in the case of functionals. These will be referred to as the H- and B-continuous functionals.

10. General forms of H-continuous and linear functionals. Remark 10.1. Let X be an H^*-normed space. By Theorem 9.1 every H-continuous functional

on X is B-continuous on X. Consequently, the general form of the H-continuous and linear functional on X must be of the same nature as but less general than the general form of the B-continuous and linear functional on X.

10.1. The space (cH). The elements of this space form a Banach space (c) when the norm of a finite complex operates as a norm for single elements. Then (cH) is an H^*-normed space. The general form of the B-continuous and linear functional for the space (c) is $f(x) = C \lim_{i=\infty} \xi_i + \sum_{i=1}^{\infty} C_i\xi_i$ (Banach, p. 66).

THEOREM 10.1. *The general form of the H-continuous and linear functional on* (cH) *is* $f(x) = C \lim_{i=\infty} \xi_i + \sum_{i=1}^{n} C_i\xi_i$.

PROOF. By Remark 10.1 every H-continuous and linear functional on (cH) is of the form of the B-continuous and linear functional on the same elements taken as a Banach space but with additional restrictions on the constant C_i's. Assume that $f(x)$ is H-continuous and linear and that there is an infinite number of non-zero constant C_i's involved in its representation. Consider the sequence $\{x_n\}$ of (cH) where $\xi_{ni} = 0$ if $n \neq i$ and $\xi_{nn} = \operatorname{sgn} 1/C_n$. This sequence is obviously such that $x_n \rightarrow_H \theta$. But for an infinite number of values of $n, f(x_n) = 1$, so $\overline{\lim}_{n=\infty} f(x_n) = 1$. Consequently, $x_n \rightarrow_H \theta$ does not imply $f(x_n) \rightarrow_B 0$. Therefore, the possibility of the general form of the H-continuous and linear functional on (cH) involving more than a finite number of non-zero constants has been eliminated. Consequently, we are confined to a consideration of the form $C \lim_{i=\infty} \xi_i + \sum_{i=1}^{n} C_i\xi_i$. This functional is readily seen to be H-continuous and linear.

10.2. The space (l^pH), $p \geqq 1$. Remarks similar to those preceding Theorem 10.1 apply here.

THEOREM 10.2. *The general form of the H-continuous and linear functional on* (l^pH), $p \geqq 1$, *is* $f(x) = \sum_{i=1}^{n} C_i\xi_i$.

PROOF. The proof of this theorem is similar to the proof of Theorem 10.1.

REMARK 10.2. If K is a finite complex then by $U(K)$ is meant the complex of the images of the elements of K by $y = U(x)$ (Vulich 1, p. 165). Vulich has proved the following theorem for operations defined on K-formed spaces: A necessary and sufficient condition that the additive operation $y = U(x)$ be KK-continuous is that there exist a constant $C > 0$ such that for every complex K of the domain $\| U(K) \| \leqq C \| K \|$. A similar statement cannot be made for HH-continuous and additive operations on H^*-normed spaces. For in the space (l^pH), $p \geqq 1$, let K be the complex (x_1, x_2) where $\xi_{pi} = 0$ if $p \neq i$ and $\xi_{pp} = 1$, and let $f(x) = \xi_1 + \xi_2$. Then $\| (x_1, x_2) \| = 0$ and $\| f(x_1), f(x_2) \| = 1$. Hence, there exists no constant $C > 0$ such that $\| f(K) \| \leqq C \| K \|$.

10.3. The space (CH). The elements of (CH) form a Banach space (C) when the norm of a finite complex operates on single elements only. Then (CH) is an H^*-normed space. The general form of the B-continuous and linear functional defined on the elements of (CH) when they are taken as the Banach space

(C) is $f(x) = \int_0^1 x(t)\, dg(t)$, where $g(t)$ is a function of bounded variation and $x(t)$ is of (C) (Banach, p. 61).

THEOREM 10.3. *The general form of the H-continuous and linear functional on* (CH) *is* $f(x) = \int_0^1 x(t)\, dg(t)$, *where* $g(t)$ *is a function of bounded variation which is constant except for a finite number of discontinuities. An alternate form is* $f(x) = \sum_{i=1}^n C_i x(t_i)$.

PROOF. It will be shown first that $g(t)$, which must be of bounded variation because of Remark 10.1, cannot be continuous and non-decreasing (or non-increasing) on $(0, 1)$ with $g(t) > g(0)$ (or $g(t) < g(0)$) for $t > 0$. Suppose that $g(t)$ is of this form. Define a sequence $\{x_n\}$ of continuous functions as follows:

$$x_n(t) = x_n(0) = 0 \text{ for } 2/n \leqq t \leqq 1,$$

$$x_n(1/n) = I^{-1}, \text{ where } I = \int_0^{1/n} t\, dg(t),^{8}$$

and $x_n(t)$ linear elsewhere. Obviously, $x_n \to_H \theta$. It will be shown that $\lim_{n\to\infty} \int_0^1 x_n(t)\, dg > 0$. Write $\int_0^1 x_n(t)\, dg = \int_0^{1/n} x_n(t)\, dg + \int_{1/n}^{2/n} x_n(t)\, dg$. On the interval $(0, 1/n)$, $x_n(t) = ntI^{-1}$. Then $\int_0^{1/n} x_n(t)\, dg = \int_0^{1/n} nI^{-1}t\, dg = nI^{-1}I = n$. On the other hand, on the interval $(1/n, 2/n)$, $x_n(t) = -nI^{-1}(t - 2/n)$, whence $\int_{1/n}^{2/n} x_n(t)\, dg = -nI^{-1}\int_{1/n}^{2/n} (t - 2/n)\, dg = -nI^{-1}\int_{1/n}^{2/n} t\, dg + 2I^{-1} \int_{1/n}^{2/n} dg$. Since in $(1/n, 2/n)$, $t \leqq 2/n$, it follows that $\int_{1/n}^{2/n} t\, dg \leqq 2/n \int_{1/n}^{2/n} dg$, and so $\int_{1/n}^{2/n} x_n(t)\, dg \geqq -2I^{-1}\int_{1/n}^{2/n} dg + 2I^{-1}\int_{1/n}^{2/n} dg = 0$. Consequently, $\int_0^1 x_n(t)\, dg \geqq n$, and therefore, if $g(t)$ is of the form specified at the beginning of this proof, $x(t)$ is of (CH), and $f(x) = \int_0^1 x(t)\, dg(t)$, then there is a sequence $\{x_n\}$ such that $x_n \to_H \theta$ but for which $\{f(x_n)\}$ does not approach zero as $n \to \infty$. Then $f(x)$ is not H-continuous. So, in general if $f(x)$ is H-continuous and linear it may not be of the form $\int_0^1 x(t)\, dg(t)$, where $g(t)$ is continuous and such that for some interval (t_0, t'), $g(t_0) < g(t)$ (or $g(t_0) > g(t)$) for $t > t_0$. Since if this were the case the sequence $\{x_n\}$ used above could be constructed on the interval (t_0, t') and then a contradiction would follow.

Therefore, $g(t)$ must be constant except for at most a denumerable number of points of discontinuity of the first kind. Assume that $g(t)$ is of this nature

[8] By the integration by parts theorem for Stieltjes integrals it is readily shown that the denominator does not vanish.

and that $f(x)$ is an H-continuous and linear functional on (CH). Let $\{a_n\}$ be a sequence of points of (0, 1) which approaches some a_0 from the right (or left) and which has the property that at each a_n the function $g(t)$ has a discontinuity of measure $b_n \neq 0$. Each a_n may itself be the limit point of points of discontinuity of $g(t)$. Let such points be designated by a_{np} and let the measure of the discontinuity of $g(t)$ at a_{np} be $b_{np} \neq 0$. About each a_n choose an interval I_n such that the left-hand end point of I_n is to the right of a_0, such that the length of $I_n \leqq \min(\text{length } I_{n-1}, 1/n)$, and such that if $a_{np} \in I_n$ then $\sum_{p=1}^{\infty} |b_{np}| \leqq |b_n|/2^n$. Define a sequence of continuous functions as follows: let $x_n(a_n) = \operatorname{sgn} 1/b_n$; on the complement of I_n let $x_n(t) = 0$, and finally let $x_n(t)$ be linear elsewhere. Then $x_n \rightarrow_H \theta$. $\int_0^1 x_n(t)\, dg = \sum_{p=1}^{\infty} [g(a_{np} + 0) - g(a_{np} - 0)]x_n(a_{np}) + b_n x_n(a_n) = \sum_{p=1}^{\infty} b_{np}x_n(a_{np}) + 1$. If each $b_{np}x_n(a_{np}) < 0$, then $\int_0^1 x_n(t)\, dg = 1 - \sum_{p=1}^{\infty} |b_{np}|\,|x_n(a_{np})| \geqq 1 - \sum_{p=1}^{\infty} |b_{np}|/2^n |b_n| > 1/2$. If each $b_{np}x_n(a_{np})$ is not less than zero then $\int_0^1 x_n(t)\, dg$ is obviously greater than 1/2. Therefore, although $x_n \rightarrow_H \theta$ it does not follow that $f(x_n) \rightarrow_B 0$. Hence, if $f(x)$ is to be H-continuous and linear it is impossible that $g(t)$ be other than constant except for at most a finite number of points of discontinuity of the first kind. In this case $f(x) = \int_0^1 x(t)\, dg = \sum_{i=1}^{n} x(t_i)[g(t_i + 0) - g(t_i - 0)] = \sum_{i=1}^{n} C_i x(t_i)$. Since only a finite number of terms are involved in this form $f(x)$ is obviously H-continuous and linear.

Remark 10.3. It is evident from this theorem that there are B-continuous and linear functionals which are not H-continuous. Therefore, the converse of Theorem 9.1 is not true.

10.4. The space (L^pH), $p \geqq 1$. Theorem 10.4. *If $y = U(x)$ is a continuous transformation from (L^pH), $p \geqq 1$, to a space Y in which the limit notion is unique, then $y = U(x)$ is a constant.*[9]

Proof. Let $x \in (L^pH)$, $p \geqq 1$, and let $y_n(t) = x(t)\cdot 1/2(1 + r(t))$, where $r_n(t)$ is the function defined in Remark 3.1. As in Remark 3.1 it is seen that $y_n \rightarrow_H \theta$ and $y_n \rightarrow_H x$. Since $y = U(x)$ is continuous it follows that $U(y_n) \rightarrow U(\theta)$ and $U(y_n) \rightarrow U(x)$. Because of the uniqueness of the limit notion in Y, $U(x) = U(\theta)$.

Corollary 10.41. *If $y = f(x)$ is an H-continuous and linear functional from (L^pH), $p \geqq 1$, then $f(x) \equiv 0$.*

11. General forms of different types of linear operations. If for the set of all continuous functions defined on (0, 1) the norm for single elements is taken to be $\max_{0 \leq t \leq 1} |x(t)|$, then the set is a Banach space and it will be denoted by (C). With respect to this norm the space (CH) is an H^*-normed space. It has been

[9] This theorem and its proof were supplied by the referee. It is a generalization of a theorem originally given by the author.

shown (Fichtenholz, p. 32) that the general form of the BB-continuous and linear operation from (C) to (C) is given by

$$(1) \qquad y(s) = U(x) = \int_0^1 x(t)\, d_t K(s, t),$$

where $K(s, t)$ is a function of two variables defined on $0 \leqq s, t \leqq 1$ and is such that

(1^0) $K(s, t)$ is of bounded variation as a function of t for each s and $\operatorname{var}_{0 \leqq t \leqq 1} K(s, t) \leqq C$, where C is a positive constant independent of s,

(2^0) $K(s, 0) = 0$; $K(s, 1)$ is continuous in s,

(3^0) $K(s, t)$ is continuous in measure[10] with respect to s in the interval $(0, 1)$ for each $s = s_0$.

If $y(s) = U(x)$ is BH-continuous and linear from (CH) to (CH), then $y(s_0) = U(x)$ is a B-continuous and linear functional defined on (CH) and it may be represented by $y(s_0) = U(x) = \int_0^1 x(t)\, d_t K(s_0, t)$, and so $y(s) = U(x)$ is of form (1) with restrictions on $K(s, t)$ which will now be examined.

THEOREM 11.1. *The class of all BH-continuous and linear operations from (CH) to (CH) is equivalent to the class of all BB-continuous and linear operations from (CH) to (CH).*

PROOF. Because of Theorem 9.2 it is evident that the class of all BB-continuous and linear operations from (CH) to (CH) is a subclass of the class of all BH-continuous and linear operations between the same spaces. Therefore, since each BH-continuous and linear operation is of the form (1), in order that an operation be BH-continuous and linear without being BB-continuous and linear it is necessary that some one of the conditions 1^0, 2^0, 3^0 on $K(s, t)$ be weakened. But these conditions are necessary and sufficient not only for the BB-continuity and linearity of $y = U(x)$ but also for the continuity of $y(s) = U(x)$. Hence, if one of these conditions is weakened $y(s)$ will no longer be continuous. Therefore, every BH-continuous and linear operation must satisfy conditions 1^0, 2^0, and 3^0 and our theorem is proved.

THEOREM 11.2. *The general form of the HH-continuous and linear operation from (CH) to (CH) is given by* (1) *where $K(s, t)$ satisfies conditions* 1^0, 2^0, 3^0 *and also*

(4^0) *for s constant $K(s, t)$ is constant in t except for at most a finite number of points of discontinuity of the first kind.*

PROOF. If $y(s) = U(x)$ is of form (1) and if $K(s, t)$ satisfies conditions 1^0–4^0, then $y(s)$ is continuous because of conditions 1^0–3^0. Moreover, if $x_n \to_H x$, then $y_n(s) \to y(s)$ for each $s \in (0, 1)$ and so $y_n \to_H y$. For consider any $s = s_0$, then $| y_n(s_0) - y(s_0) | \leqq \left| \int_0^1 [x_n(t) - x(t)]\, d_t K(s_0, t) \right|$. The last expression is the

[10] $K(s, t)$ is said to be convergent in measure with respect to s for $s = s_0$ on $0 \leqq t \leqq 1$ if $K(s, t)$ as a function of t approaches $K(s_0, t)$ in measure when $s \to s_0$.

absolute value of an H-continuous and linear functional (see Theorem 10.4 and condition 4^0), and therefore, $x_n \rightarrow_H x$ implies $y_n(s_0) \rightarrow y(s_0)$. This proves our contention.

If $y(s) = U(x)$ is an HH-continuous and linear operation from (CH) to (CH), then it is of form (1) and $K(s, t)$ satisfies 1^0–4^0. For consider any $s = s_0$, then $y(s_0) = U(x)$ is merely an H-continuous and linear functional on (CH). Hence, $y(s_0) = \int_0^1 x(t)\, d_tK(s_0, t)$ and $K(s_0, t)$ must satisfy 4^0 because of the restriction of Theorem 10.4. Then $y(s) = \int_0^1 x(t)\, d_tK(s, t)$. Since $y(s) = U(x)$ is HH-continuous and linear and (CH) is an H^*-normed space, it follows from Theorem 9.1 that $y(s)$ is BH-continuous and linear. By Theorem 11.1, $y(s) = U(x)$ is also BB-continuous and linear and so $K(s, t)$ must satisfy conditions 1^0–3^0.

THEOREM 11.3. *The general form of the HB-continuous and linear operation from* (CH) *to* (CH) *is given by* (1) *where* $K(s, t)$ *satisfies conditions* 1^0–3^0 *and also*

(5^0) *there exists at most a finite number of points* $t_1, t_2, \cdots, t_q$ *such that for any value* $s = s_0$ *the points of discontinuity of* $K(s_0, t)$ *are to be found among the points* $t_i (i = 1, 2, \cdots, q)$.

PROOF. As in the proof of Theorem 11.2 it is readily seen that any HB-continuous and linear operation is of the form (1). If $y = U(x)$ is HB-continuous and linear, then by Theorem 9.3 it is HH-continuous and linear, and so by Theorem 11.2 $K(s, t)$ satisfies conditions 1^0–3^0. Suppose that 5^0 is not satisfied by $K(s, t)$. Then there exists a sequence $\{s_n\}$ of points of the interval (0, 1) such that the points of discontinuity of the first kind which are contributed by $\{K(s_n, t)\}$ are infinite in number, that is to say, there is an infinite sequence $\{t_\nu\}$ of points such that each is a point of discontinuity of some $K(s_n, t)$. Choose a subsequence $\{t_{\nu_p}\}$ of $\{t_\nu\}$ such that $\{t_{\nu_p}\}$ approaches some t_0 from the right (or left). About each t_{ν_p} as center and associated with each $K(s_n, t)$ there is an interval $I(t_{\nu_p}, s_n)$ such that within it there is no point of discontinuity of $K(s_n, t)$ except perhaps t_{ν_p} itself. The existence of this interval is assured by the HH-continuity of each HB-continuous operation. Let I_{ν_p} be an interval $I(t_{\nu_p}, s_n)$ which does not contain t_0 and whose right-hand end point is to the left of $t_{\nu_p} + (t_{\nu_p} - t_0)/2$ and which is such that the measure $a^n_{\nu_p}$ of the discontinuity of $K(s_n, t)$ at t_{ν_p} is not equal to zero. Define a sequence $\{x_n\}$ of continuous functions as follows:

$$x_n(t_{\nu_p}) = \text{sgn } 1/a^n_{\nu_p},$$

$$x_n(t) = 0 \text{ for } t \text{ not of } I_{\nu_p},$$

and $x_n(t)$ linear elsewhere. Then $x_n \rightarrow_H \theta$. Moreover, $y_n(s) = \int_0^1 x_n(t)\, d_tK(s, t)$ is such that $| y_n(s_n) | = | x_n(t_{\nu_p}) | \cdot | K(s_n, t_{\nu_p} + 0) - K(s_n, t_{\nu_p} - 0) | = | \text{sgn } 1/a^n_{\nu_p} | \cdot | a^n_{\nu_p} | = 1$. Therefore, $\| y_n \| = \max_{0 \leq s \leq 1} | y_n(s) | = 1$, and so $x_n \rightarrow_H \theta$ does not imply $y_n \rightarrow_B \theta$. This is a contradiction and so 5^0 must hold.

Let $y(s) = U(x)$ be of the form (1) and be defined on (CH), and let $K(s, t)$ satisfy conditions 1^0, 2^0, 3^0, and 5^0. Because of conditions 1^0–3^0, $y(s)$ is continuous. Moreover, $y(s) = \int_0^1 x(t)\, d_tK(s, t) = \sum_{i=1}^q x(t_i)[K(s, t_i + 0) - K(s, t_i - 0)] = \sum_{i=1}^q x(t_i)C_i(s)$ by condition 5^0. From 1^0 we know that there exists a constant $C > 0$ such that $| C_i(s) | \leqq C$ for each i and for $s \in (0, 1)$. Then $| y_n(s) - y(s) | \leqq C \sum_{i=1}^q | x_n(t_i) - x(t_i) |$. If $x_n \rightarrow_H x$, then since only a finite number of points are involved in the last summation, $|| y_n - y || \rightarrow 0$.

PHILADELPHIA COLLEGE OF PHARMACY

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CHARACTERIZATION OF COMPLEX COUPLE SPACES

BY ARISTOTLE D. MICHAL AND MAX WYMAN

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Introduction. In considering generalisations of classical complex analysis for complex Banach spaces it seems desirable to consider complex couple spaces. For such spaces many of the classical theorems can be shown to hold.[1] The following paper attempts to characterize such spaces. This is done by means of a conjugate operation.

1. Characterization of Hermitian couple spaces.

DEFINITION 1.1. Two complex Banach spaces B_1, B_2 shall be called equivalent[2] if there exists a single valued function $f(z)$ on B_1 to B_2 with the properties

(a) $f(z)$ is linear[3] in z, and is homogeneous of degree one,

(b) $f(z)$ ranges over the whole of B_2, and is solvable in z,

(c) $\| f(z) \| = \| z \|$.

If conditions (a), (b) are satisfied we say B_1, B_2 are linearly homeomorphic.

Let E be a real Banach space with a real inner product $[x, y]$.[4] From E we can construct a complex Banach space $E(c)$ as follows. Let $E(c)$ be the set of all couples $\{x, y\}$ where x, y are elements of E. We define

$$\{x_1, y_1\} = \{x_2, y_2\} \quad \text{if and only if} \quad x_1 = x_2, y_1 = y_2, \tag{1}$$

$$\{x_1, y_1\} + \{x_2, y_2\} = \{x_1 + x_2, y_1 + y_2\}, \tag{2}$$

$$(a + ib)\{x, y\} = \{ax - by, bx + ay\} \quad \text{where } a, b \text{ are real numbers,} \tag{3}$$

$$\| \{x, y\} \| = (\| x \|^2 + \| y \|^2)^{\frac{1}{2}} \tag{4}$$

DEFINITION 1.2. A complex Banach space B is called a Hermitian couple space, if it is equivalent to a space of type $E(c)$.

DEFINITION 1.3. Let B be a complex Banach space. A bilinear function $[Z, U]$ on B^2 to the complex numbers is called a Hermitian inner product[5] if

(i) $\overline{[Z, U]} = [U, Z]$, $[(a + ib)Z, U] = (a + ib)[Z, U]$,

[1] For work in general analysis see Taylor (3), Michal and Martin (4), and Michal, Davis and Wyman (6).

[2] For the notion of equivalence of real Banach spaces see Banach (1), p. 180.

[3] We use the term linear to mean additive and continuous. In real Banach spaces this is enough to ensure homogeneity of degree one. This is no longer so in complex Banach spaces.

[4] For postulates of a real inner product see Michal (5). In addition we make the restriction that $\| x \| = [x, x]^{\frac{1}{2}}$.

[5] These are of course the same postulates given for an inner product in Hilbert space. See Stone (2) and the references to Von Neumann's papers given there.

(ii) $[Z, Z] \geqq 0$ and $[Z, Z] = 0$ if and only if $Z = 0$,
(iii) $\| Z \| = [Z, Z]^{\frac{1}{2}}$.

THEOREM 1. *A necessary and sufficient condition that an arbitrary complex Banach space B be a Hermitian couple space is that*
(α) *B must possess a Hermitian inner product* $[Z, U]$
(β) *there exist a function* $\bar{Z}$[6] *on B to B such that*

$$(\beta_1) \quad \overline{[Z_1, Z_2]} = [\bar{Z}_1, \bar{Z}_2],$$

$$(\beta_2) \quad \bar{\bar{Z}} = Z.$$

Proof of necessity. For a space $E(c)$ of couples $z = \{x, y\}$ the existence of a Hermitian inner product[7] $[z, u]$ was shown in Michal, Davis and Wyman (6). The function $\bar{z}$ for such spaces is the "ordinary complex" conjugate $\bar{z} = \{x, -y\}$. Let $Z = f(z)$ be the mapping function which makes B equivalent to $E(c)$. We define
(a) $[Z, U] = [f^{-1}(Z), f^{-1}(U)]$, where the inner product on the right is that of $E(c)$,
(b) $\bar{Z} = f(\bar{z})$.

By straightforward calculation $[Z, U]$ is a Hermitian inner product for B, and condition (β) is satisfied. To prove the sufficiency we make use of the following lemmas.

LEMMA 1. *Let B be a complex Banach space satisfying* (α), (β) *of theorem* 1. *Then* $\bar{Z}$ *is linear in Z, and*
(1.1) $\overline{(a + ib)Z} = (a - ib)\bar{Z}$, *for a, b real.*
For all Z_1, Z_2, Z_3 we have

$$(1.2) \quad \overline{[\overline{Z_1 + Z_2}, Z_3]} = [Z_1 + Z_2, \bar{Z}_3] = [\bar{Z}_1, Z_3] + [\bar{Z}_2, Z_3] = [\bar{Z}_1 + \bar{Z}_2, Z_3].$$

Thus $\overline{Z_1 + Z_2} = \bar{Z}_1 + \bar{Z}_2$, and $\bar{Z}$ is additive in Z. From $\| \bar{Z} \| = \| Z \|$ we can conclude that $\bar{Z}$ is linear in Z. Similarly property (1.1) is shown.

LEMMA 2. *The totality E of all elements W for which* $\bar{W} = W$ *is a real sub-Banach space of B and is such that* $\| W \|$ *is generated by a real symmetric inner product.*

Clearly under the operations of B, E is a real normed linear space. If $\{W_i\}$ is a Cauchy convergent sequence of E there is an element W of B such that $\| W - W_n \| < \frac{\epsilon}{2}$. From $\| \bar{Z} \| = \| Z \|$ we obtain $\| \bar{W} - W_n \| < \frac{\epsilon}{2}$, and hence $\| W - \bar{W} \| < \epsilon$. Thus $W = \bar{W}$ and W is an element of E. Thus E is a complete space, and hence is a real Banach space.

[6] We have used the notation $\bar{Z}$ as a function on B to B. This should not be confused with the complex conjugate notation of ordinary complex numbers. See (β_1) where both occur.

[7] If x, y, ζ, η are elements of E, and $z = \{x, y\}$, $u = \{\zeta, \eta\}$ are in $E(c)$, then the Hermitian inner product $[z, u]$ of $E(c)$ is given by $[z, u] = [x, \zeta] + [y, \eta] + i\{[y, \zeta] - [x, \eta]\}$, where $[x, y]$ is the inner product of E.

For any W_1, W_2 of E we have

$$\overline{[W_1, W_2]} = [\overline{W}_1, \overline{W}_2] = [W_1, W_2] = [W_2, W_1]. \tag{1.3}$$

This completes the proof of lemma 2.

From the E of lemma 2, let us construct the Hermitian couple space $E(c)$. For any element Z of B, $\left\{\dfrac{Z + \overline{Z}}{2}, \dfrac{i\overline{Z} - iZ}{2}\right\}$ is an element of $E(c)$. Then B is equivalent to $E(c)$ and the required mapping function is

$$f(Z) = \left\{\frac{Z + \overline{Z}}{2}, \frac{i\overline{Z} - iZ}{2}\right\} \tag{1.4}$$

The additivity and homogeneity of degree one of $f(Z)$ are easy to verify. From $\| Z \|^2 = [Z, Z]$ we readily obtain that

$$\| f(Z) \| = \left(\frac{\| Z + \overline{Z} \|^2}{4} + \frac{\| i\overline{Z} - iZ \|^2}{4}\right)^{\frac{1}{2}} = \| Z \|.$$

Finally the equation $f(Z) = \{W_1, W_2\}$ has the unique solution $Z = W_1 + iW_2$ for any W_1, W_2 of E. Thus B is a Hermitian couple space.

2. Characterization of complex couple spaces. If E is an arbitrary Banach space, the couple space $E(c)$ is a complex Banach space under definitions (1)–(4) if and only if the norm of E is generated by a real inner product. In a previous paper (Michal, Davis and Wyman (6)) the authors raised the question of giving a definition to replace (4) which would eliminate the restriction on the norm of E. A definition was communicated to us by E. W. Paxson which could be used if E were separable. Later on A. E. Taylor[8] gave us verbally a definition which places no restriction on E. This result is stated in the following theorem.

THEOREM 2. *Let E be an arbitrary real Banach space, and let $x, y \in E$. The totality $E'(c)$ of couples $\{x, y\}$ form a complex Banach space under definitions* (1), (2), (3) *and*

$$\| \{x, y\} \| = \sup_{\text{mod } l=1} (l^2(x) + l^2(y))^{\frac{1}{2}}.\text{[9]} \tag{4'}$$

The truth of this theorem is easily verified. Complex spaces which are linearly homeomorphic to spaces of the type $E'(c)$ are called complex couple spaces.

THEOREM 3. *A necessary and sufficient condition that an arbitrary complex Banach space B be a complex couple space is that there exist a function $\overline{Z}$ on B to B with the properties*:

$$\text{I} \qquad \overline{Z_1 + Z_2} = \overline{Z}_1 + \overline{Z}_2,$$

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[8] Both Taylor and Paxson have seen the manuscript of Michal, Davis and Wyman (6). Taylor had also seen Paxson's result.

[9] $l(x)$ means a linear functional on E to the real numbers. The superior is taken over all linear functionals whose modulus is one.

II $$\overline{iZ} = -i\overline{Z},$$

III $$\overline{\overline{Z}} = Z,$$

IV $$\|\overline{Z}\| = \|Z\|.$$

The proof follows essentially the same lines as the proof of theorem 1. $\overline{Z}$ will now trivially satisfy lemma 1, and the first part of lemma 2 will be satisfied. From E we construct $E'(c)$ and the latter will be linearly homeomorphic to B. The continuity of the mapping function $f(Z) = \left\{\dfrac{Z + \overline{Z}}{2}, \dfrac{i\overline{Z} - iZ}{2}\right\}$ follows from $\|f(Z)\| \leqq \sqrt{2}\,\|Z\|$.

California Institute of Technology
Pasadena, California.

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SEQUENCES DEFINED BY NON-LINEAR ALGEBRAIC DIFFERENCE EQUATIONS[1]

BY OTIS E. LANCASTER

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An algebraic difference equation

$$\Phi(y(x + m), y(x + m - 1), \cdots, y(x), x) = 0, \tag{1}$$

Φ a polynomial with rational coefficients in its arguments $x, y(x), y(x + 1), \cdots, y(x + m)$, defines a sequence[2]

$$y(a), y(a + 1), \cdots, y(a + m - 1), y(a + m), \cdots$$

when the values of $y(x)$ are assigned at the m points $a, a + 1, \cdots, a + m - 1$. Obviously every equation (1) defines an infinite number of sequences: one for each set of values $y(a), y(a + 1), \cdots, y(a + m - 1)$.

We propose to obtain some of the properties of such sequences by studying the difference equations which define them. In the study of infinite sequences one is mainly interested in their ultimate behavior. Hence we are here interested in the behavior of the solutions of the difference equations, for integral values of x, in the neighborhood of infinity. Although, at present, it is not possible to write out explicitly the solutions of most non-linear difference equations, it seems that it might be feasible to determine whether a given equation defines sequences that approach zero as x becomes infinite by considering the solutions of the linear difference equation formed by omitting the non-linear terms. (We assume that at least one linear term appears.) For when $y(x)$ approaches zero the linear terms are infinitesimals of the first order, while the non-linear terms are infinitesimals of higher order. Therefore, it would be expected that the behavior of such sequences is largely determined by the linear terms of the difference equation. With this idea in mind, we attempt to gain information about the sequences defined by a non-linear difference equation by considering the solution of the difference equation formed from its linear terms.[3]

It is clear that once we have criteria for determining when a difference equation defines sequences which approach zero we can easily determine whether one defines sequences that approach a constant limit α. For, if after the application of the transformation $y(x) = \bar{y}(x) + \alpha$, the transformed difference equa-

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[1] The author's thanks are due to Professor G. D. Birkhoff for his counsel during the preparation of this paper.

[2] When the equation (1) is non-linear in the variable $y(x + m)$ some convention must be made as to the one of the values of $y(x + m)$ that is to be taken at each step.

[3] The investigations of Lattès (1), Horn (2) and Perron (3) have also had as their basis the same idea.

tion defines sequences that approach zero, then the original equation defines sequences that approach α.

Besides determining conditions sufficient to insure that the sequences of a difference equation converge to a definite limit, we study the effect of the degree of the equation, of the order of the equation, and of variable coefficients upon the rate of convergence of the sequences. Lastly we determine difference equations whose rational sequences converge most rapidly to irrational numbers of the form $\beta^{1/p}$, where p is an integer and β is a rational number.

For the discussion we write (1) in the form

$$\Phi(S_{n+m}, S_{n+m-1}, \cdots, S_n, n) = 0, \tag{2}$$

where S_{n+i} denotes the value of $y(n + i)$ when n is an integral value of x. The sequences are then denoted by $\{S_n\}$. Of course, when we speak of the solutions of the equation (2) it is understood that n is a continuous real variable, and $y(n) = S_n$.

1. Sequences that approach zero.

DEFINITION. The phrase "*the difference equation of the linear terms*" shall denote the difference equation formed from the given equation by omitting all terms that are of higher than the first degree in the arguments $S_n, S_{n+1}, \cdots, S_{n+m}$, where it is understood that the linear equation is to be of the *same order* as that of the non-linear equation.

For example, the difference equation of the linear terms of the non-linear equation,

$$S_{n+2}S_nS_{n+1} + nS_nS_{n+2} + S_n + nS_{n+1} + n^2 = 0$$

is

$$0 \cdot S_{n+2} + nS_{n+1} + S_n = -n^2.$$

The characteristic equation of this linear equation is

$$0 \cdot \rho^2 + 1\rho + 0 = 0.$$

It follows from the above definition that besides the usual possibilities for infinite and zero roots of the characteristic equation, the characteristic equation of the equation of the linear terms always has an infinite root if S_{n+m} does not appear linearly in the non-linear equation and a zero root if S_n does not appear linearly.

THEOREM 1. *If in a non-linear algebraic difference equation a coefficient of a linear term is of as high degree in n as the coefficients of the non-linear terms, and if there are solutions of the difference equation of linear terms which approach zero as n becomes infinite, and if the characteristic equation of the difference equation of linear terms has finite, non-zero roots not equal to one in absolute value, then there are sequences defined by the given equation that approach zero as n becomes positively infinite.*

The truth of this theorem follows immediately from the following theorem of Horn[(2)]. "In the system of non-linear difference equations

$$(3) \qquad y_i(x+1) = G_i(x, y_1(x), \cdots, y_m(x)), \qquad (i = 1, 2, \cdots, m)$$

let G_i be a function of $x, y_1(x), \cdots, y_m(x)$, which vanishes at the point $x = \infty$, $y_1 = 0, \cdots, y_m = 0$ and is analytic in a neighborhood of it:

$$G_i(x, y_1(x), \cdots, y_m(x)) = \frac{a_{i0}}{x} + a_{i1}y_1 + \cdots + a_{im}y_m + \cdots \qquad (i = 1, \cdots, m).$$

If $s - a_1, \cdots, s - a_m$ are the elementary divisors of the determinant

$$| a_{ij} - s\delta_{ij} |,$$

let the system be transformed to the system

$$(4) \qquad y_i(x+1) = a_i y_i(x) + f_i(1/x, y_1(x), \cdots, y_m(x)) \qquad (i = 1, \cdots, m)$$

where

$$f_i(1/x, y_1, \cdots, y_m) = \sum A^{(i)}_{\lambda, \lambda_1, \lambda_2, \cdots, \lambda_m} (1/x)^{\lambda} y_1^{\lambda_1} \cdots y_m^{\lambda_m}$$

$$(\lambda = 1, \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0 : \lambda + \lambda_1 + \cdots + \lambda_m \geqq 2)$$

and

$$0 < | a_1 | \leqq | a_2 | \leqq \cdots \leqq | a_\mu | < 1 < | a_{\mu+1} | \leqq \cdots \leqq | a_m |.$$

If it is possible to choose h so small and x_0 so large that

$$| f_i(1/x, y_1, \cdots, y_m) | < hq$$

and

$$| f_i(1/x, y_1', \cdots, y_m') - f_i(1/x, y_1, \cdots, y_m) |$$

$$(5) \qquad \leqq g(| y_1' - y_1 | + \cdots + | y_m' - y_m |) \text{ for } x > x_0, \; | y_i | \leqq h, \; | y_1' | \leqq h$$

$$(i = 1, 2, \cdots, m),$$

where $q \leqq \frac{1}{2} | a_i |^{1/2}(1 - | a_i |^{1/2})$, $(i = 1, 2, \cdots, \mu)$; $q \leqq | a_i |^{1/2}(| a_i |^{1/2} - 1)$ $(i = \mu + 1, \cdots, m)$, $g < q/m$ and if $\eta_i \leqq h/2$, *then* there exists a unique solution[4] $y_i(x)$, $(i = 1, \cdots, m)$ of the system for which $| y_i | \leqq h$ for $x > x_0$, $y_i(x_0) = \eta_i$ $(i = 1, 2, \cdots, \mu)$ and $\lim_{x \to \infty} y_i(x) = 0$."[5]

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[4] In order to obtain a unique solution Horn admitted only constants for the arbitrary periodic functions of period one. The theorem holds when any periodic functions are admitted if the η_i are defined over the interval $(x_0, x_0 + 1)$.

[5] Although the theorem above is "translated" directly from Horn's paper (2), we see that if two or more of the a's are equal then the system (3) cannot in general be brought to the form (4). Since, however, an obvious extension of his work shows that the theorem holds for the case of equal a's, we assume the theorem for the general case. If some of the a's are equal the equations (4) are of the form

$$y_i(x+1) = a_i y_i + k y_{i-1}(x) + f(1/x, y_1, \cdots, y_m)$$

when $a_i = a_{i-1}$.

We prove Theorem 1 by showing that the conditions of Horn's theorem are all fulfilled. Since, by hypothesis, a solution of the equation of the linear terms approaches zero as $n \to \infty$, $\Phi(S_{n+m}, \cdots, S_n, n)$ vanishes at the point $n = \infty$, $S_n = 0, \cdots, S_{n+m} = 0$. Hence (2) may be written in the form

$$\begin{aligned} a_m S_{n+m} + a_{m-1}S_{n+m-1} + \cdots + a_0 S_n \\ + \Phi_1(S_{n+m}, S_{n+m-1}, \cdots, S_n, n) + a_{-1}n^{-1} = 0, \end{aligned} \tag{6}$$

where Φ_1 is a polynomial whose terms are of higher than first degree in the arguments $1/n$, $S_n, \cdots, S_{n+m}$. It follows from the assumption that the roots of the characteristic equation of the equation of the linear terms are *finite*, that $a_m \neq 0$. And when $a_m \neq 0$, (2) treated as an algebraic equation in S_{n+m} has a solution

$$S_{n+m} = f(S_{n+m-1}, \cdots, S_n, n) \tag{7}$$

which vanishes at the point $S_{n+m-1} = 0, \cdots, S_n = 0$, $n = \infty$. That is, the expansion of f in terms of $1/n$, $S_n, \cdots, S_{n+m-1}$ does not contain a constant term.

We show that equation (7) defines sequences that approach zero, from which it follows that equation (2) defines sequences which approach zero. Replacing S_{n+m} in Φ_1 by f, we have (7) in the form

$$\begin{aligned} S_{n+m} = -\frac{1}{a_m}[a_0 S_n + a_1 S_{n+1} + \cdots + a_{m-1}S_{n+m-1} \\ + \Phi_1(f, S_{n+m-1}, \cdots, S_n, n) + a_{-1}n^{-1}]; \end{aligned} \tag{8}$$

where the terms in the expansion of Φ_1 are of higher than first degree in $1/n$, $S_n, \cdots, S_{n+m-1}$. Upon setting $u_1(n) = S_n$, $u_2(n) = S_{n+1}, \cdots, u_m(n) = S_{n+m-1}$, we obtain from the equation (8) the system of difference equations

$$\begin{aligned} u_i(n+1) &= u_{i+1}(n) \qquad (i = 1, 2, \cdots, m-1) \\ u_m(n+1) &= -\frac{1}{a_m}[a_0 u_1(n) + \cdots + a_{m-1}u_m(n) \\ &\qquad + \Phi_1(f, u_m(n), \cdots, u_1(n), n) + a_{-1}n^{-1}]. \end{aligned} \tag{9}$$

The auxiliary equation of this system,

$$\begin{vmatrix} -\rho & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\rho & 1 \\ \frac{a_0}{a_m} & \frac{a_1}{a_m} & \frac{a_2}{a_m} & \frac{a_3}{a_m} & \cdots & \frac{a_{m-2}}{a_m} & \frac{a_{m-1}}{a_m} + \rho \end{vmatrix} = 0, \tag{10a}$$

is equivalent to the characteristic equation of the equation of the linear terms, viz.

$$a_m\rho^m + a_{m-1}\rho^{m-1} + \cdots + a_0 = 0. \tag{10}$$

When the roots of (10) are distinct, we transform the system (9) into the canonical form

$$z_i(n + 1) = \rho_i z_i(n) + \Psi_i(1/n, z_1(n), \cdots, z_m(n)) \qquad (i = 1, 2, \cdots, m) \tag{11}$$

and when (10) has equal roots, into one of the same form, except that when $a_j = a_{j-1}$ the j^{th} equation is

$$z_j(n + 1) = \rho_j z_j(n) + z_{j-1}(n) + \Psi_j(1/n, z_1(n), \cdots, z_m(n)).$$

The Ψ_i $(i = 1, \cdots, m)$ satisfy the condition (5), since the terms in their expansions, except the one a_{-1}/n, are of higher than first degree in the variables $1/n$, $z_1(n), \cdots, z_m(n)$. Therefore, there are sequences $z_i(n)$ which approach zero as $n \to \infty$, and consequently, sequences $\{S_n\}$ defined by (7), that approach zero. Q.E.D.

It follows from the proof of Horn's theorem that an algebraic difference equation (linear or non-linear) cannot have more than one solution that approaches zero as $n \to \infty$ if the roots of the characteristic equation of the difference equation of the linear terms are greater than one in absolute value. Hence if there is only one solution of the difference equation of the linear terms that approaches zero as n becomes infinite there is at most one solution of the non-linear difference equation that approaches zero. However, if there are an infinite number of solutions of the difference equation of the linear terms which approach zero and if the roots of the characteristic equation are not equal to one in absolute value, then there are an infinite number of sequences of (7) that approach zero. Of course, it is clear that if a sequence is to approach zero the initial conditions S_{n_0+i} $(i = 0, \cdots, m - 1)$, which determine it, must be chosen sufficiently small. Moreover, *if only one of the roots of the characteristic equation* (10) *is less than one in absolute value, then only one of these initial conditions is arbitrary.* If two roots are less than one in absolute value then two conditions are arbitrary, etc. If all the roots are less than one in absolute value, then all sequences defined by (7) converge to zero, for an arbitrary choice of initial conditions in the region $|S_{n_0+i}| \leqq h/2$, $(i = 1, \cdots, m - 1)$. We call such a region a region of strong convergence.

When there is only one sequence defined by a non-linear difference equation that has a zero limit, it may be the trivial one $S_n = 0$ for all n. In fact, it is always such a sequence unless the equation has a term of zero$^{\text{th}}$ degree in the arguments $S_n, S_{n+1}, \cdots, S_{n+m}$. *In the future we exclude such sequences.*

The hypotheses of Theorem 1 exclude all the difference equations which do not contain both the linear terms a_0S_n and a_mS_{n+m}, $(a_0 \neq 0,\ a_m \neq 0)$. We have the following result concerning those in which $a_0 = 0$.

THEOREM 2. *If in a non-linear algebraic difference equation a coefficient of a linear term is of higher degree in n than the coefficients of the non-linear terms, and all solutions of the difference equation of the linear terms approach zero as n becomes infinite, then there are sequences defined by the given equation that approach zero as* $n \to \infty$.

PROOF. When all solutions of the difference equation of the linear terms approach zero all roots of the characteristic equation must be less than one in absolute value, hence $a_m \neq 0$. Some or all of the other a_i's may be zero. If $a_0 \neq 0$, this theorem is just a special case of Theorem 1: hence we assume that $a_0 = 0$. That is, we assume that the difference equation has the form

$$(12)\quad \begin{aligned} a_m S_{n+m} + a_{m-1} S_{n+m-1} + \cdots + a_k S_{n+k} + \phi_1(S_{n+m}, \cdots, S_n, n) \\ + a_{-1} n^{-1} = 0, \qquad k > 0. \end{aligned}$$

We replace S_{n+m} in Φ_1 by one of its values

$$S_{n+m} = f(S_{n+m-1}, \cdots, S_n, n),$$

which vanishes at the point $1/n = 0$, $S_{n+m-1} = 0, \cdots, S_n = 0$, and then set $u_1(n) = S_n, \cdots, u_m(n) = S_{n+m-1}$. This gives the system of equations

$$(13)\quad \begin{aligned} u_i(n+1) &= u_{i+1}(n) \qquad (i + 1, 2, \cdots, m-1) \\ u_m(n+1) &= -\frac{1}{a_m}[a_k u_{k+1}(n) + \cdots + a_{m-1} u_m \\ &\quad + \Phi_1(f, u_1(n), \cdots u_m(n), n) + a_{-1} n^{-1}] \end{aligned}$$

When the non-zero roots of the characteristic equation of the equation of the linear terms of (12) are distinct, we apply the linear transformation

$$(14)\quad \begin{aligned} u_i(n) &= z_i(n) \qquad (i = 1, 2, \cdots, k) \\ u_i(n) &= a_{i,k+1} z_{k+1}(n) + a_{i,k+2} z_{k+2}(n) + \cdots + a_{i,m} z_m(n) \\ &\qquad (i = k+1, \cdots, m) \end{aligned}$$

where the a_{ij} $(i, j = k+1, \cdots, m)$ are so determined that the transformed system of difference equations has the form

$$(15)\quad \begin{aligned} z_i(n+1) &= z_{i+1}(n) \qquad (i = 1, 2, \cdots, k) \\ z_i(n+1) &= \rho_i z_i(n) + \Psi_i(z_1(n), \cdots, z_m(n), n) \qquad (i = k+1, \cdots, m). \end{aligned}$$

In order to clarify the proof we divide it into two parts. First we assume that the Ψ_i do not involve n explicitly. Under this assumption, for a given $\delta > 0$, however small, it is possible to choose h so small that

$$|\Psi_i(z_1(n), \cdots, z_m(n))| < \delta\{|z_1| + \cdots + |z_m|\} \qquad (i = k+1, \cdots, m)$$

when $|z_i| \leqq h$ $(i = 1, 2, \cdots, m)$. Therefore

$$|z_i(n+1)| < |\rho_i||z_i(n)| + \delta\{|z_1(n)| + \cdots + |z_m(n)|\} \leqq (|\rho_i| + m\delta)h$$
$$(i = k+1, \cdots, m).$$

If $|\rho_j| \geqq |\rho_i|$, $(j \neq i)$, we set $|\rho_j| + m\delta = \sigma_1$ and choose δ so small that $\sigma_1 < 1$. Then,

$$|z_i(n+1)| < \sigma_1 h < h \qquad (i = k+1, \cdots, m).$$

It follows from the equations (15) that

$$|z_i(n+1)| \leqq h \qquad (i = 1, 2, \cdots, k),$$

and

$$|z_k(n+2)| < \sigma_1 h,$$

hence

$$|z_i(n+2)| < \sigma_1 h \qquad (i = k, \cdots, m)$$
$$|z_i(n+2)| < h \qquad (i = 1, 2, \cdots, k-1),$$

and so on until

$$|z_i(n+k+1)| < \sigma_1 h \qquad (i = 1, \cdots, m).$$

In general

$$|z_i(n+lk+1)| < \sigma_1^l h \qquad (i = 1, \cdots, m).$$

Hence the $z_i(n + lk)$ $(i = 1, \cdots, m)$ approach zero as l becomes infinite, consequently, there are sequences of (12) which approach zero as n becomes infinite.

Now if n appears explicitly in the Ψ_i, for a given $\delta > 0$, we may choose h so small and n_0 so large that

$$(16) \qquad |\Psi_i| < \delta(|z_1(n)| + \cdots + |z_m|) + M/n \qquad (i = k+1, \cdots, m)$$

for $|z_i| < h$ $(i = 1, \cdots, m)$ and $n \geqq n_0$. Hence for the initial conditions

$$|z_i(n_0)| < h,$$

$$|z_i(n_0+1)| < |\rho_i||z_i(n_0)| + \delta(|z_1(n_0)| + \cdots + |z_m(n_0)|) + M/n_0$$
$$< (|\rho_i| + m\delta)h + M/n_0 \qquad (i = k+1, \cdots, m).$$

If $|\rho_j| \geqq |\rho_i|$ for $i \neq j$, we set $|\rho_j| + m\delta = \sigma_1$ and choose δ so small that $\sigma_1 < 1$. After n_0 and h have been determined so that (16) is satisfied, we increase n_0 until $\sigma_1 + M/hn_0 = \sigma_2 < 1$. Then

$$|z_i(n_0+1)| < \sigma_1 h + M/n_0 < \sigma_2 h < h \qquad (i = k+1, \cdots, m)$$
$$|z_i(n_0+1)| < h \qquad (i = 1, \cdots, k)$$

and

$$|z_i(n_0+k+1)| < \sigma_1 h + M/n_0 < \sigma_2 h \qquad (i = 1, \cdots, m).$$

The z_i's may never exceed h in absolute value, and when l is so large that

$$\frac{M}{h\sigma_2(n_0 + l)} < \sigma_2 - \sigma_1$$

we have

$$|z_i(n_0 + 2k + l + 1)| < |\rho_i||z_i| + \delta(|z_i| + \cdots + |z_m|) + \frac{M}{n_0 + l} < \sigma_2^2 h \qquad (i = 1, \cdots, m).$$

Continuing step by step we see that the z_i $(i = 1, \cdots, m)$ are bounded by an expression which slowly approaches zero as n becomes infinite: hence they approach zero.

When the characteristic equation has equal roots, we employ a transformation (14) in which the a_{ij} $(i, j = k + 1, \cdots, m)$ are so determined that when $\rho_l = \rho_{l-1}$ the l^{th} equation of the transformed system is of the form

$$z_l(n + 1) = \rho_l z_l(n) + \epsilon z_{l-1}(n) + \Psi_l(z_1, z_2, \cdots, z_m, n), \tag{17}$$

where ϵ is chosen so small that $|\rho_j| + \epsilon < 1$, $|\rho_j| \geqq |\rho_i|$, $i \neq j$, and the remaining equations are of the form (11). Treating this new system by a method analogous to the one used above, in which $|\rho_j| + \epsilon$ plays the role corresponding to that of $|\rho_j|$, we see that it has solutions which approach zero. Hence our proof is complete.

Theorems 1 and 2 exclude difference equations in which the degree of n in the coefficient of the non-linear terms is higher than the degree of n in the coefficients of the linear terms. In regard to such equations we have the following theorem.

THEOREM 3. *An algebraic difference equation, whose terms are of at least first degree in the S_{n+i}, defines sequences which approach zero as n becomes infinite, if all roots of the characteristic equation of the equation of the linear terms are less than one in absolute value*

PROOF. As in the proof of Theorems 1 and 2, we replace S_{n+m} by one of its values that vanishes at the point $S_n = 0, \cdots, S_{n+m-1} = 0$, and then we apply a linear transformation which transforms this new equation into a system of equations of the form (11), (15) or (17).

CASE I. The coefficients of the linear terms in the S_{n+i} are constants.

When all roots of the characteristic equation are distinct and different from zero, the system has the form

$$z_i(n + 1) = \rho_i z_i(n) + \Psi_i(z_1(n), \cdots, z_m(n), n) \qquad (i = 1, \cdots, m),$$

where all terms of the Ψ_i $(i = 1, \cdots, m)$ are of at least second degree in the z_i. Hence there exists an h such that

$$|\Psi_i(z_1, \cdots, z_m, n)| < Mn^l(|z_1| + \cdots + |z_m|)^2 \qquad (i = 1, \cdots, m)$$

when $|z_i| < h$ $(i = 1, \cdots, m)$, where M and l are constants. If $|\rho_j| \geqq |\rho_i|$

for all i, we choose a number σ such that $|\rho_j| < \sigma < 1$, and then choose n_0 so large that

$$|m^2 M n_0^l \sigma^{n_0}| + |\rho_j| \leqq \sigma.$$

Such a choice of n_0 is possible since $\lim_{n\to\infty} n^l \alpha^n = 0$, if $|\alpha| < 1$. Now for z_i in the region $|z_i| \leqq h \leqq \sigma^{n_0}$ $(i = 1, \cdots, m)$

$$\begin{aligned}
|z_i(n_0+1)| &\leqq |\rho_i||z_i| + M n_0^l m^2 \sigma^{2n_0} \\
&\leqq (|\rho_i| + M n_0^l m^2 \sigma^{n_0})\sigma^{n_0} \leqq \sigma^{n_0+1} \qquad (i = 1, 2, \cdots, m) \\
|z_i(n_0+2)| &\leqq |\rho_i||z_i(n_0+1)| + M(n_0+1)^l m^2 \sigma^{2(n_0+1)} \\
&\leqq (|\rho_i| + M(n_0+1)^l m^2 \sigma^{n_0+1})\sigma^{n_0+1} \\
&< (|\rho_i| + M n_0^l m^2 \sigma^{n_0})\sigma^{n_0+1} < \sigma^{n_0+2} \qquad (i = 1, \cdots, m),
\end{aligned}$$

and in general

$$|z_i(n_0+p)| < \sigma^{n_0+p} \qquad (i = 1, 2, \cdots, m).$$

Hence the $z_i(n_0 + p)$ approach zero as p becomes infinite, and consequently $\{S_n\} \to 0$ as $n \to \infty$.

When some of the ρ_i equal zero the transformed system of equations is of the form (15). Again we choose σ such that $|\rho_j| < \sigma < 1$, but now we take n_0 so large that

$$m^2 M(\mu n_0)^l \sigma^{n_0} + |\rho_j| \leqq \sigma, \text{ where } \mu \text{ is the maximum of } k \text{ and } 2.$$

Then for $|z_i(n_0)| \leqq \sigma^{n_0}$ $(i = 1, \cdots, m)$

$$\begin{aligned}
|z_i(n_0+1)| &\leqq |\rho_i||z_i| + M n_0^l m^2 \sigma^{2n_0} \\
&\leqq (|\rho_j| + M n_0^l m^2 \sigma^{n_0})\sigma^{n_0} \leqq \sigma^{n_0+1} \qquad (i = k+1, \cdots, m) \\
|z_i(n_0+1)| &\leqq \sigma^{n_0} \qquad (i = 1, 2, \cdots, k): \\
|z_i(n_0+2)| &\leqq |\rho_j||z_i| + M(n_0+1)^l m^2 \sigma^{2n_0+2} \\
&\leqq \sigma^{n_0}(|\rho_j| + M m^2 (\mu n_0)^l \sigma^{n_0}) \leqq \sigma^{n_0+1} \qquad (i = k, \cdots, m) \\
|z_i(n_0+2)| &\leqq \sigma^{n_0} \qquad (i = 1, 2, \cdots, k-1):
\end{aligned}$$

and

$$\begin{aligned}
|z_i(n_0+k+1)| &\leqq |\rho_j||z_i| + M(n_0+k)^l m^2 \sigma^{2(n_0+k)} \\
&\leqq (|\rho_j| + M n^2 (\mu n_0)^l \sigma^{n_0})\sigma^{n_0} \leqq \sigma^{n_0+1} \qquad (i = 1, \cdots, m).
\end{aligned}$$

Consequently,

$$|z_i(n_0+k+2)| \leqq |\rho_j||z_i| + M(n_0+k+1)^l m^2 \sigma^{2(n_0+k+1)} \leqq \sigma^{n_0+2}$$

$$(i = k+1, \cdots, m)$$

$$| z_i(n_0 + 2k + 1) | \leqq \sigma^{n_0+2} \qquad (i = 1, \cdots, m).$$

Continuing step by step we have

$$| z_i(n_0 + pk + 1) | \leqq \sigma^{n_0+p} \qquad (i = 1, \cdots, m).$$

Hence the $z_i(n_0 + pk + 1)$ approach zero as $p \to \infty$, and the corresponding sequence $\{S_n\} \to 0$ as $n \to \infty$.

As in the proof of Theorem 2, the above treatment, with suitable modifications, applies to the case where the characteristic equation has multiple roots.

Case II. The coefficients of the linear terms in S_{n+i} are polynomials in n. The transformed system of equations may be written in the form

$$\begin{aligned} z_i(n+1) &= z_{i+1}(n) \qquad (i = 1, \cdots, k) \\ z_i(n+1) &= \rho_i z_i(n) + b_{i1}(1/n) z_1(n) + \cdots + b_{im}(1/n) z_m(n) \\ &\quad + \Psi_i(z_1(n), \cdots, z_m(n), n) \qquad (i = k+1, \cdots, m), \end{aligned} \tag{18}$$

where the b_{ij} are convergent series in $1/n$, whose constant terms are zero, and all terms of the Ψ_i $(i = k + 1, \cdots, m)$ are of at least second degree in the arguments z_i $(i = 1, \cdots, m)$. We first choose n_1 so large that

$$| \rho_i | + mb = \rho < 1$$

where $b \geqq | b_{ij}(1/n) |$ $(i, j = 1, \cdots, m)$ for $n > n_1$. Then the argument in case I holds where $| \rho_i |$ is replaced by ρ and n_0 is chosen greater than n_1. Hence our proof is complete.

2. **Sequences with finite limits.** The only possible finite limits for the sequences defined by an algebraic difference equation, with constant coefficients,

$$\Phi(S_{n+m}, S_{n+m-1}, \cdots, S_n) = 0 \tag{19'}$$

are the roots of the algebraic equation

$$\Phi(S, S, \cdots, S) = 0, \tag{20'}$$

where the latter is obtained from the former by replacing S_{n+i} $(i = 1, \cdots, m)$ by S. Likewise, the only possible finite limits for the sequences defined by an algebraic difference equation

$$\Phi_1(S_{n+m}, \cdots, S_n, n) = 0 \tag{19}$$

are the roots of the equation

$$\Phi_1'(S, \cdots, S) = 0, \tag{20}$$

where $\Phi_1' = \lim_{n\to\infty} n^{-q} \Phi_1(S_{n+m}, \cdots, S_n, n)$ and q is the highest power of n appearing in Φ_1.

If $S = \alpha$ is a root of the equation (20), and if the difference equation

$$\bar{\Phi}(\bar{S}_{n+m}, \cdots, \bar{S}_n, n) = 0, \tag{21}$$

obtained from (19) by the transformation $S_n = \bar{S}_n + \alpha$, satisfies the hypothesis of theorems 1, 2 or 3, then (19) defines sequences which approach α. For the sake of brevity, we shall call the difference equation (21) the "*transformed equation.*"

THEOREM 4. *If α is an algebraic number, then there exist algebraic difference equations that define sequences which approach α.*[6]

To construct one: it is sufficient to take an algebraic difference equation, involving arbitrary parameters in such a manner that the equation (20) has a root α, and then to choose values for the parameters so that there are sequences defined by the transformed equation (21) which approach zero.

To illustrate the method, we construct a difference equation of second order that defines sequences which approach $\beta^{\frac{1}{2}}$, where β is a rational number. The difference equation

$$aS_nS_{n+1} + bS_{n+1}S_{n+2} - cS_{n+1}^2 - (b + a - c)\beta = 0 \tag{22}$$

may define sequences which approach $\beta^{\frac{1}{2}}$ or $-\beta^{\frac{1}{2}}$, since its limit as $n \to \infty$ is $S^2 = \beta$. Applying the transformation $S_n = \bar{S}_n + \beta$, we obtain for the equation of the linear terms of the transformed equation

$$b\bar{S}_{n+2} + (b + a - 2c)\bar{S}_{n+1} + a\bar{S}_n = 0. \tag{23}$$

If we choose a, b and c such that the roots of the characteristic equation of (23) are less than one in absolute value, there are sequences of (22) which converge to $\beta^{\frac{1}{2}}$. For example when $a = 1$, $b = 6$, $c = 6$, the roots of the characteristic equation (23) are 1/2 and 1/3, hence some sequences of (21) will converge to $\beta^{\frac{1}{2}}$. In particular, the difference equation

$$S_nS_{n+1} + 6S_{n+1}S_{n+2} - 6S_{n+1}^2 - 2 = 0 \tag{24}$$

defines sequences which approach $\sqrt{2}$. For the initial values $S_0 = 1$, $S_1 = 2$ and $S_0 = 1.4$, $S_1 = 1.4$ the sequences are

$$1, 2, 7/6, 9/7, 1021/756, \cdots;$$

and

$$1.4, 1.4, 59/42 = 1.404+, 34908/24780$$
$$= 1.4087+, 4804998967/3395410200 = 1.4122+, \cdots$$

respectively. If instead of the above values we take $a = 1$, $b = 54$, $c = 35$, $\beta = 2$, we obtain the difference equation

$$54S_{n+1}S_{n+2} - 35S_{n+1}^2 + S_{n+1}S_n - 40 = 0. \tag{25}$$

[6] The truth of this theorem follows immediately from the work of Schröder (4) on the approximation of the roots of an algebraic equation by the iteration of rational functions.

For the initial conditions $S_0 = 1$, $S_1 = 2$ and $S_0 = 1.4$, $S_1 = 1.4$ the sequences are

$$1,\ 2,\ 23/38,\ 30647/22356 = 1.326+,\ 1.4052+,\ \cdots$$

and

$$1.4,\ 1.4,\ 1333/985 = 1.4105+,\ 48074279/34011495 = 1.4134+,\ \cdots$$

respectively.

The sequences defined by (25) converge more rapidly to the $\sqrt{2}$ than the corresponding sequences defined by (24). The question arises, is there a difference equation whose sequences approach a given algebraic number more rapidly than those defined by all other difference equations? Under the conditions for which the generalized Poincare theorem[3] holds, the smaller the absolute values of the roots—all greater than zero—of the characteristic equation of the equation of the linear terms the more rapid the convergence to zero of the solution of the difference equation. Hence we were led to predict the following theorem.

THEOREM 5. *Of the family of non-linear difference equations of the form*

$$a_m(n)S_{n+m} + a_{m-1}(n)S_{n+m-1} + \cdots + a_0(n)S_n + \Phi(S_{n+m}, \cdots, S_n, n) = \theta(n); \tag{26}$$

where Φ is a fixed polynomial whose terms are of at least second degree in the S_{n+i}, the $a_i(n)$ $(i = 0, 1, \cdots, m)$ and $\theta(n)$ are polynomials in n, $a_m(n) \not\equiv 0$, and the coefficients of $a_m(n)$ are fixed while those of the $a_i(n)$ $(i = 0, 1, \cdots, m-1)$ and $\theta(n)$ are arbitrary; the one that defines sequences which converge most rapidly to zero, for arbitrary initial conditions from a sufficiently small region surrounding the origin, is the one in which the $a_i(n)$, $(i = 0, 1, \cdots, m-1)$ and $\theta(n)$ are identically equal to zero.

PROOF. (a) First, $\theta(n)$ must vanish. When $\theta(n) \not\equiv 0$ the sequences cannot approach zero more rapidly than n^{-k}, where k is an integer. For if this were true, the left member of (26) would approach zero more rapidly than the right member which is impossible. The proof of Theorem 3 reveals that there are equations of form (26) which define sequences that approach zero as rapidly as $Mn^l\sigma^n$. Therefore if the sequences have the maximum rate of convergence, $\theta(n) \equiv 0$.

(b) Second, the degree of n in $a_m(n)$ must be greater than or equal to the degree of n in all the other a's. Otherwise, there is no region of strong convergence.

(c) Third, if the coefficients of the terms of highest degree in n of $a_0(n), \cdots, a_m(n)$ are $a_{00}, a_{10}, \cdots, a_{m0}$, respectively, then the difference equation of the family whose sequences converge most rapidly to zero is the one for which $a_{m0} \neq 0$ and $a_{i0} = 0$, $(i = 0, 1, \cdots, m-1)$.

To prove this we refer back to the notation and the statements in the proof of Theorem 3. When there is one a_{i0} that is different from zero there is at least one ρ_i that is not equal to zero. It is only necessary to consider the case where all of the ρ_i are less than one in absolute value. For it is possible to show, by

a slight alteration of the inequalities in the proof of Theorem 3, that of the sequences defined by two difference equations, identical except for the ρ_i of maximum absolute value, those defined by the difference equation with the smaller ρ_i converge more rapidly than *some* of those defined by the other equation.

So we assume that $1 > |\rho_j| > |\rho_i|$ $(i \neq j)$, and consider the equations (18). We choose n_0 so large that

$$m^2 M(\mu n)^l \sigma^n < \tfrac{1}{2}(|\rho_j|/3)^{m+1}$$

$$< \frac{|\rho_j| - |\rho_i|}{2} \qquad (|\rho_l| \neq |\rho_r|; l, r = 1, \cdots, m)$$

$$< \epsilon/2 \qquad \text{when } \rho_l = \rho_{l-1},$$

where ϵ is the ϵ of equation (17);

$$m\,|b_{ij}(n)| \leqq mb \leqq \tfrac{1}{2}(|\rho_j|/3)^{m+1};$$

$$\rho_j + mb = \rho < 1;$$

and

$$m^2M(\mu n)^l\sigma^n + \rho < \sigma < 1, \qquad n \geqq n_0 .$$

When the initial conditions are $|z_i(n_0)| \leqq \sigma^{n_0}$, the last two conditions are sufficient to insure that the sequences approach zero such that

$$|z_i(n_0 + pk + 1)| < \sigma^{n_0+p}.$$

When $z_j(n_0) = \sigma^{n_0}$ and $|z_i(n_0)| \leqq \sigma^{n_0}$, $(i = 1, 2, \cdots, m)$ we have in succession:

$$|z_j(n_0 + 1)| \geqq |z_j(n_0)|\,|\rho_j| - b\sum_{r=1}^{m-1}|z_r(n_0)| - M(n_0)^l\left[\sum_{r=1}^{m-1}|z_r(n_0)|\right]^2$$

$$\geqq |z_j(n_0)|\,[|\rho_j| - (|\rho_j|/3)^{m+1}] \geqq z_j(n_0)(2|\rho_j|/3),$$

$$|z_j(n_0 + 2)| \geqq |z_j(n_0 + 1)|\,|\rho_j|$$

$$- b\sum|z_r(n_0 + 1)| - M(n_0 + 1)^l[\sum|z_r(n_0 + 1)|]^2$$

$$\geqq |z_j(n_0 + 1)|\left[|\rho_j| - \frac{\max|z_r(n_0 + 1)|}{|z_r(n_0 + 1)|}\cdot\left(\frac{|\rho_j|}{3}\right)^{m+1}\right]$$

$$\geqq |z_j(n_0 + 1)|\left[|\rho_j| - \left(\frac{|\rho_j|}{3}\right)^m\right] \geqq |z_j(n_0 + 1)|\cdot\left(\frac{2|\rho_j|}{3}\right),$$

....

$$|z_j(n_0 + l)| \geqq |z_j(n_0)|\left(\frac{2|\rho_j|}{3}\right)^l, \qquad (l = 2, 3, \cdots, m + 1)$$

Under the above restrictions,

$$\frac{|z_i(n)|}{|z_j(n)|} \leqq \left(\frac{2|\rho_j|}{3}\right)^{-m}, \qquad n \geqq n_0$$

Thus,

$$
\begin{aligned}
&|z_j(n_0 + m + 2)| \\
&\quad \geqq |z_j(n_0 + m + 1)| \left[|\rho_j| - \{b - M(n_0 + m + 1)^l \sum |z_r(n_0 + m + 1)|\} \cdot \frac{\sum |z_r(n_0 + m + 1)|}{|z_j(n_0 + m + 1)|} \right] \\
&\quad \geqq |z_j(n_0 + m + 1)| \left[|\rho_j| - (|\rho_j|/3)^{m+1} \left(\frac{2|\rho_j|}{3} \right)^{-m} \right] \\
&\quad \geqq |z_j(n_0 + m + 1)| \left(\frac{2|\rho_j|}{3} \right) \\
&\quad \geqq z_j(n_0)(2|\rho_j|/3)^{m+2}.
\end{aligned}
$$

Continuing, we obtain

$$|z_j(n_0 + m + l)| \geqq z_j(n_0)(2|\rho_j|/3)^{l+m} \qquad l = 0, 1, \cdots.$$

Therefore, if some ρ_i is different from zero, *some* sequences defined by the difference equation converge to zero more slowly than the sequence $(2 |\rho_j|/3)^n$. While if $\rho_i = 0$, $i = 1, 2, \cdots, m$, all sequences, with initial conditions $|z_i(n_0)| \leqq \sigma^{n_0}$, converge to zero more rapidly than the sequence $(|\rho_j|/3)$. Thus for the most rapid convergence of all sequences, with arbitrary initial conditions, $a_{i0} = 0$ $(i = 0, 1, \cdots, m - 1)$.

(d) Fourth, the $a_i(1/n) \equiv 0$ $(i = 0, 1, \cdots, m - 1)$. For assume this statement is false and set $S_n = n^{\lambda n} T_n$, where λ is so determined that in the new equation one other linear term is of the same degree in n as $a_m(n)$. The quantity λ will be negative. Remove the common factor $n^{\lambda n}$. Now, although this new equation is not an algebraic equation, the inequalities are stronger than they were for the proof of (c) above. Therefore the argument of (c) applies. Hence by a repetition of the argument if the sequences have the maximum rate of convergence $a_i(1/n) \equiv 0$ $(i = 0, 1, \cdots, m - 1)$. Q.E.D.

Let us again consider the example. According to Theorem 5 there is a difference equation of the form (22) that defines sequences which converge more rapidly to $\sqrt{2}$ than those defined by (25). It is the one for which $a = 0$, $b + a - 2c = 0$. That is, the equation

$$2S_{n+1}S_{n+2} - S_{n+1}^2 - 2 = 0,$$

or

$$2S_nS_{n+1} - S_n^2 - 2 = 0. \tag{27}$$

For the initial values $S_0 = 1$ and $S_0 = 1.4$ the sequences are

$$1, 1.5, 1.4166+, 577/408 = 1.41421\overset{*}{5}+, 665857/470832 = 1.41421356237\overset{*}{6}+ \cdots$$

$$886731088897/627013566048 = 1.4142135623730950 \cdots +, \cdots$$

an

$$7/5 = 1.4,\ 99/70 = 1.4142\overset{*}{8}+,\ 19601/13860 = 1.41421356\overset{*}{4},$$

$$768398401/543338720 = 1.41421356237309504\overset{*}{9}9+$$

respectively. The asterisk indicates the first incorrect decimal of the terms. The sequences defined by this equation converge to the limit very rapidly. If we assume the error of the n^{th} term is ϵ and set $S_n = 2^{\frac{1}{2}} + \epsilon$, then

$$S_{n+1} = 2^{\frac{1}{2}} + 2^{-\frac{3}{2}}\epsilon^2 - 2^{-2}\epsilon^3 + \cdots.$$

Hence, when S_n is sufficiently close to the limit the error is approximately squared with each successive term of the sequence. We say that the order of rate of convergence of the sequences is 2.

The difference equation (27) is of first order, and we know from the work of Schröder[4] on iteration of rational functions that it is possible to set up a first order difference equation with constant coefficients, whose sequences approach a given algebraic number α with an *order of rate of convergence* k, where k is any integer. That is, there exists a first order difference equation with the property that when S_n is replaced by $\alpha + \epsilon$, then

$$S_{n+1} = \alpha + c_1\epsilon^k + c_2\epsilon^{k+1} + \cdots,$$

where c_1 is a constant independent of ϵ. (If $k = 1$, $| c_1 | < 1$).

Is this true for difference equations of higher than first order?

THEOREM 6. *For a given integer k and an algebraic number α, there exist algebraic difference equations of the m^{th} order that define rational sequences which approach α with an order of rate of convergence k.*

PROOF. Let

$$S_{n+m} = \Phi(S_n, \cdots, S_{n+m-1}, n) \tag{28}$$

be an algebraic difference equation, where Φ, a rational function in its arguments $S_n, \cdots, S_{n+m-1}, n$, has the Taylor's expansion

$$\begin{aligned}\Phi = a_0 + a_{10}(S_n - \alpha) + a_{11}(S_{n+1} - \alpha) + \cdots + a_{1m-1}(S_{n+m-1} - \alpha)\\ + a_{200}(S_n - \alpha)^2 + a_{210}(S_{n+1} - \alpha)(S_n - \alpha) + \cdots\end{aligned} \tag{29}$$

which converges in some neighborhood of the point $S_n = \alpha, \cdots, S_{n+m-1} = \alpha$, $n = \infty$. (The a's are rational functions of n.)

There are sequences defined by (28) which converge to α with an order of rate of convergence k, if, when $S_n = \alpha + \epsilon$, $S_{n+1} = \alpha + \theta_1\epsilon^k, \cdots, S_{n+m-1} = \alpha + \theta_{m-1}\epsilon^{k^{m-1}}$, where the θ_i $(i = 1, \cdots, m-1)$ are constants independent of ϵ, then $S_{n+m} = \alpha + \theta_m\epsilon^{k^m} + c_2\epsilon^{k^m+1} + \cdots$. This is true, when $a_0 = \alpha$ and $a_{li_1,\cdots,i_l} \equiv 0$ for all values of l for which

$$k^{i_1} + k^{i_2} + \cdots + k^{i_l} < k^m \qquad (i_1, i_2, \cdots, i_l < m) \tag{30}$$

and when $a_{li_1,\cdots,i_l} \neq 0$ for some values of $i_1, \cdots, i_l$ such that $k^{i_1} + k^{i_2} + \cdots + k^{i_l} = k^m$. Hence, if there exists a rational function Φ such that $\Phi(\alpha, \cdots, \alpha, n) \equiv \alpha$ and

$$\left.\frac{\partial^{j_1+j_2+\cdots+j_m}\Phi}{\partial S_n^{j_1}\,\partial S_{n+1}^{j_2}\cdots\partial S_{n+m-1}^{j_m}}\right|_{\alpha} \equiv 0$$

for all j's such that

$$j_1 + kj_2 + k^2j_3 + \cdots + k^{m-1}j_m < k^m$$

then the theorem is true.

We state that if α is a simple root of an algebraic equation $f(z) = 0$, then

$$\begin{aligned}
&\Phi(S_n, S_{n+1}, \cdots, S_{n+m-1}, n)\\
&\quad= \frac{1}{m}\left[S_n + \sum_{j=1}^{k^m-1}\left\{(-1)^j\frac{f^j(S_n)}{j!}\left(\frac{1}{f'(S_n)}\frac{d}{dS_n}\right)^{j-1}\frac{1}{f'(S_n)}\right\} + f^{k^m}(S_n)\phi_0\left(S_n, \frac{1}{n}\right)\right]\\
&\qquad+ \frac{1}{m}\left[S_{n+1} + \sum_{j=1}^{k^{m-1}-1}\left\{(-1)^j\frac{f^j(S_{n+1})}{j!}\left(\frac{1}{f'(S_{n+1})}\frac{d}{dS_{n+1}}\right)^{j-1}\frac{1}{f'(S_{n+1})}\right\}\right.\\
&\qquad\qquad\left.+ f^{k^{m-1}}(S_{n+1})\phi_1\left(S_{n+1}, \frac{1}{n}\right)\right]\\
&\qquad+ \cdots + \frac{1}{m}\left[S_{n+m-1} + \sum_{j=1}^{k-1}\left\{(-1)^j\frac{f^j(S_{n+m-1})}{j!}\left(\frac{1}{f'(S_{n+m-1})}\frac{d}{dS_{n+m-1}}\right)^{j-1}\right.\right.\\
&\qquad\qquad\left.\left.\cdot\frac{1}{f'(S_{n+m-1})}\right\} + f^k(S_{n+m-1})\phi_{m-1}\left(S_{n+m-1}, \frac{1}{n}\right)\right]
\end{aligned}$$

is a function[7] satisfying the above conditions, where the ϕ_i $(i = 0, 1, \cdots, m-1)$ are arbitrary polynomials which are finite at the point $S_{n+i} = \alpha$ $(i = 1, \cdots, m-1)$ and $n = \infty$, f' is the first derivative of f with respect to its variable, f^j is the j^{th} power of f, and $((1/f'(z))\, d/dz)^j$ is a symbolic operator which subjects $1/f'(z)$ to the operation of differentiation and multiplication j times. For example

$$\left(\frac{1}{f'}\frac{d}{dz}\right)^4\frac{1}{f'} = \frac{1}{f'}\frac{d}{dz}\left(\frac{1}{f'}\frac{d}{dz}\left\{\frac{1}{f'}\frac{d}{dz}\left[\frac{1}{f'}\frac{d}{dz}\left(\frac{1}{f'}\right)\right]\right\}\right).$$

First, $\Phi(\alpha, \alpha, \cdots, \alpha, n) = \alpha$, for when $S_{n+i} = \alpha$, $(i = 1, \cdots, m-1)$, all terms, except the first in each of the brackets, contain a factor $f(\alpha)$ and hence vanish. Second, the $k^m - 1$ partial derivatives with respect to S_n vanish at the point $S_{n+i} = \alpha$, $(i = 1, \cdots, m-1)$, since the derivatives of the expressions in the last $m - 1$ brackets are zero and Schröder has proved that the $k^m - 1$ derivatives of the expression in the first bracket vanish when $S_n = \alpha$. Similarly, the $k^{m-i} - 1$ partial derivatives of Φ with respect to S_{n+i} vanish when

[7] This is a generalization of a formula developed by Schröder.

$S_{n+1} = \alpha$. And lastly, all second partial derivatives with respect to two different S_{n+i} vanish identically. Q.E.D.

ASSUMPTION. In order to assure that the sequences under consideration are rational we shall assume, in the future, that S_{n+m} enters the equation only in the first degree.

3. The effect of the order of a difference equation and the degree of n in the coefficients upon the order of rate of convergence of the sequences. It follows from the work of Schröder, and from the above theorem, that it is possible to increase the order of rate of convergence of sequences, defined by difference equations, by increasing the degree of the equations. Now we ask the questions: When the degree is held constant, how does a change in the order of the difference equation affect the order of the convergence of the sequences? What is the relation between the degree of n in the coefficients of a difference equation and the order of convergence of the sequences?

As a partial answer to the last question we have the following:

THEOREM 7. *The order of rate of convergence, r, of rational sequences defined by a first order difference equation whose coefficients are polynomials in n, cannot exceed the order of convergence of rational sequences defined by* SOME *first order difference equation of the same degree with constant coefficients, provided $r \geqq 2$.*

PROOF. Again without loss in generality, we consider only sequences which approach zero. Thus reducing our problem to that of showing that, "if S_n is a null sequence defined by a difference equation of order one and degree q, then the order of rate of convergence r of S_n is $<q + 1$ and it is equal to q for some equations of this type which have constant coefficients."

When the coefficients are constants, the maximum rate of convergence of the sequences defined by first order difference equations of the q^{th} degree is q. If the sequences do converge to zero with this maximum rate of convergence then the Taylor's expansion of the left member of the equation

$$S_{n+1} = \Phi(S_n),$$

gives

$$S_{n+1} = a_q S_n^q + a_{q+1} S_n^{q+1} + \cdots$$

where $a_q \neq 0$. If the coefficients of Φ were polynomials in n then the a_i would be rational functions of n. In order for the sequences of such an equation to have the maximum rate of convergence $a_i \equiv 0$, $(i = 1, \cdots, q - 1)$; $a_q(n) \not\equiv 0$, and $a_q(n)$ must be a rational function of n in which the degree of the numerator is less than the degree of the denominator, viz.

$$a_q(n) = a_\lambda n^{-\lambda} + a_{\lambda+1} n^{-\lambda-1} + \cdots \qquad (\lambda > 0),$$

and the equation may be written in the form

$$S_{n+1} = n^{-\lambda} S_n^q[(a_\lambda + a_{\lambda+1} n^{-1} + \cdots) + n^\lambda a_{q+1}(n) S_n + n^\lambda a_{q+2}(n) S_n^q + \cdots]. \tag{31}$$

Hence if the order of rate of convergence of the sequences may be increased by allowing the coefficients to be polynomials in n, the order of convergence of the sequences of (31) must be greater than that of the sequences defined by the equation

$$S_{n+1} = a_\lambda S_n^q .$$

That is, the order of rate of convergence must equal or exceed the order of convergence of the sequences of

$$T_{n+1} = a_\lambda T_n^{q+1}. \tag{32}$$

Now when the sequences converge with an order of rate $\geqq 2$, we may choose n_0 so large and then S_{n_0} so small that

$$a_\lambda^{-1} \mid a_{\lambda+1}n^{-1} + a_{\lambda+2}n^{-2} + \cdots \mid < \epsilon/2 < \tfrac{1}{2}, \qquad n > n_0 ,$$

$$n^\lambda a_\lambda^{-1} \mid a_{q+1}(n)S_n + a_{q+2}(n)S_n^2 + \cdots \mid < \epsilon/2 < \tfrac{1}{2}, \qquad n > n_0 ,$$

with such a choice the sequences defined by (31) are term by term greater in absolute value than the sequences defined by the difference equation

$$\bar{S}_{n+1} = n^{-\lambda} a_\lambda (1 - \epsilon) \bar{S}_n^q . \tag{31$'$}$$

The solution of this equation for $\bar{S}_{n_0} = S_{n_0}$ is

$$\bar{S}_{n_0+p} = S_{n_0}^{q^p} a_\lambda^{(q^p-1)/q} (1 - \epsilon)^{(q^p-1)/q} (p + n_0 - 1)^{-\lambda} (p + n_0 - 2)^{-\lambda q} \cdots (n_0)^{-\lambda q^{p-1}}$$

and the solution of (32) is

$$T_{n_0+p} = a_\lambda^{((q+1)^p-1)/q} S_{n_0}^{(q+1)^p}.$$

But

$$\lim_{p\to\infty} \left(\frac{T_{n_0+p}}{\bar{S}_{n_0+p}}\right) = \lim_{p\to\infty} (S_{n_0})^{(q+1)^p - q^p} a_\lambda^{((q+1)^p-q^p)/q} (1 - \epsilon)^{(q^p-1)/q} (p + n_0 - 1)^\lambda \cdot (p + n_0 + 2)^{\lambda q} \cdots (n_0)^{\lambda q^{p-1}} = 0.$$

Therefore the sequences of (31$'$), and consequently those of (31), do not converge to zero as rapidly as those of (32), so they cannot have an order of rate of convergence $q + 1$. Q.E.D.

Although we have been unable to generalize Theorem 7 we should like to conjecture that it holds for m^{th} order equations.

Now let us concentrate upon the relation between the order of difference equations and the order of the rate of convergence of their sequences.

THEOREM 8. *There exist algebraic difference equations of higher than first order, that define rational sequences which converge to certain algebraic numbers with a higher order of convergence than the sequences defined by any first order difference equations of the same degree.*

PROOF. Take the general equation of the m^{th} order and q^{th} degree with constant coefficients,

$$\sum_{j=1}^{q} \sum_{i_1=0}^{m-1} \sum_{i_2 \geqq i_1}^{m-1} \cdots \sum_{i_j \geqq i_{j-1}}^{m} (c_{0i_1i_2\cdots i_j} S_{n+i_1} S_{n+i_2} \cdots S_{n+i_j}) + c_0 = 0,$$

or

$$(33)\qquad S_{n+m} = -\frac{\sum_{j=1}^{q}\sum_{i_1=0}^{m-1}\sum_{i_2\geq i_1}^{m-1}\cdots\sum_{i_j\geq i_{j-1}}^{m-1}(c_{0i_1i_2\cdots i_j}S_{n+i_1}S_{n+i_2}\cdots S_{n+i_j})+c_0}{\sum_{j=1}^{q}\sum_{i_1=0}^{m-1}\sum_{i_2\geq i_1}^{m-1}\cdots\sum_{i_{j-1}\geq i_{j-2}}^{m-1}(c_{0i_1i_2\cdots i_j}S_{n+i_1}S_{n+i_2}\cdots S_{n+i_j})+c_{0m}}$$

and demand that it define sequences which approach the number $\beta^{1/q}$ with an order of rate of convergence k, where k is an integer to be determined, β is a rational number, and $\beta^{j/q}$ is irrational ($j = 1, \cdots, q-1$). That is, set $S_{n+j} = \beta + \theta_j\epsilon^{k^j}$ ($j = 0, 1, \cdots, m-1$), expand the right member of (33) in powers of ϵ and demand that the constant term equal $\beta^{1/q}$, and that the coefficients of ϵ^t vanish for $t = 1, 2, \cdots, k^m - 1$. These demands impose k^m linear homogeneous relations upon the coefficients of the given equation (33); as can readily be seen by dividing the numerator by the denominator. The k^m linear relations may contain some of the θ_i ($i = 0, \cdots, m-1$) and some fractional powers of β. In order to insure that the coefficients of the difference equation (33) be rational, the coefficients of θ_i and the fractional power of β in these linear relations must vanish.

The maximum value for k is determined by the number of the relations that the coefficients of (33) may satisfy, that is, by the number of coefficients. This number[8] is

$$\sum_{j=1}^{q}\frac{m(m+1)\cdots(m+j-2)(m+2j-1)}{j!}+1.$$

Since, when $m = 1$ it is $\sum_{j=1}^{q}[2] + 1 = 2q + 1$, the coefficients of a first order difference equation may be chosen so as to satisfy at least $2q$ linear homogeneous relations. Therefore k may be at least 2, for when $k = 2$ the coefficients are only restricted by $2q$ relations, q relations which insure that the constant term in the power series in ϵ equal $\beta^{1/q}$ and q relations which insure that the coefficient of ϵ equal zero. Hence if a difference equation of higher than first order defines sequences whose order of rate of convergence, k, is greater than that of sequences defined by a first order difference equation, k must be at least 3.

[8] In order to see this, write

$$\sum_{i_1=0}^{m-1}\cdots\sum_{i_j\geq i_{j-1}}^{m}=\sum_{i_1=0}^{m-1}\cdots\sum_{i_j\geq i_{j-1}}^{m-1}+\sum_{i_1=0}^{m-1}\cdots\sum_{i_{j-1}\geq i_{j-2}}^{m-1}$$

The number of terms in the first sum on the right is $m(m+1)\cdots(m+j-1)/j!$. Therefore, the total number of terms of j^{th} degree is

$$\frac{m(m+1)\cdots(m+j-1)}{j!}+\frac{m(m+1)\cdots(m+j-2)}{(j-1)!}$$
$$=\frac{m(m+1)\cdots(m+j-2)(m+2j-1)}{j!}.$$

Hence if the order of rate of convergence of the sequences may be increased by allowing the coefficients to be polynomials in n, the order of convergence of the sequences of (31) must be greater than that of the sequences defined by the equation

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$$T_{n+1} = a_\lambda T_n^{q+1}. \tag{32}$$

Now when the sequences converge with an order of rate $\geqq 2$, we may choose n_0 so large and then S_{n_0} so small that

$$a_\lambda^{-1} \mid a_{\lambda+1}n^{-1} + a_{\lambda+2}n^{-2} + \cdots \mid < \epsilon/2 < \tfrac{1}{2}, \qquad n > n_0 ,$$

$$n^\lambda a_\lambda^{-1} \mid a_{q+1}(n)S_n + a_{q+2}(n)S_n^2 + \cdots \mid < \epsilon/2 < \tfrac{1}{2}, \qquad n > n_0 ,$$

with such a choice the sequences defined by (31) are term by term greater in absolute value than the sequences defined by the difference equation

$$\bar{S}_{n+1} = n^{-\lambda}a_\lambda(1 - \epsilon)\bar{S}_n^q . \tag{31$'$}$$

The solution of this equation for $\bar{S}_{n_0} = S_{n_0}$ is

$$\bar{S}_{n_0+p} = S_{n_0}^{q^p}a_\lambda^{(q^p-1)/q}(1 - \epsilon)^{(q^p-1)/q}(p + n_0 - 1)^{-\lambda}(p + n_0 - 2)^{-\lambda q} \cdots (n_0)^{-\lambda q^{p-1}}$$

and the solution of (32) is

$$T_{n_0+p} = a_\lambda^{((q+1)^p-1)/q} S_{n_0}^{(q+1)^p}.$$

But

$$\lim_{p\to\infty}\left(\frac{T_{n_0+p}}{\bar{S}_{n_0+p}}\right) = \lim_{p\to\infty} (S_{n_0})^{(q+1)^p-q^p}a_\lambda^{((q+1)^p-q^p)/q}(1 - \epsilon)^{(q^p-1)/q}(p + n_0 -1)^\lambda$$
$$\cdot (p + n_0 + 2)^{\lambda q} \cdots (n_0)^{\lambda q^{p-1}} = 0.$$

Therefore the sequences of (31′), and consequently those of (31), do not converge to zero as rapidly as those of (32), so they cannot have an order of rate of convergence $q + 1$. Q.E.D.

Although we have been unable to generalize Theorem 7 we should like to conjecture that it holds for m^{th} order equations.

Now let us concentrate upon the relation between the order of difference equations and the order of the rate of convergence of their sequences.

THEOREM 8. *There exist algebraic difference equations of higher than first order, that define rational sequences which converge to certain algebraic numbers with a higher order of convergence than the sequences defined by any first order difference equations of the same degree.*

PROOF. Take the general equation of the m^{th} order and q^{th} degree with constant coefficients,

$$\sum_{j=1}^{q}\sum_{i_1=0}^{m-1}\sum_{i_2\geqq i_1}^{m-1}\cdots\sum_{i_j\geqq i_{j-1}}^{m}(c_{0i_1i_2\cdots i_j}S_{n+i_1}S_{n+i_2}\cdots S_{n+i_j}) + c_0 = 0,$$

or

$$(33)\qquad S_{n+m} = -\frac{\sum_{j=1}^{q}\sum_{i_1=0}^{m-1}\sum_{i_2\geqq i_1}^{m-1}\cdots\sum_{i_j\geqq i_{j-1}}^{m-1}(c_{0i_1i_2\cdots i_j}S_{n+i_1}S_{n+i_2}\cdots S_{n+i_j}) + c_0}{\sum_{j=1}^{q}\sum_{i_1=0}^{m-1}\sum_{i_2\geqq i_1}^{m-1}\cdots\sum_{i_{j-1}\geqq i_{j-2}}^{m-1}(c_{0i_1i_2\cdots i_j}S_{n+i_1}S_{n+i_2}\cdots S_{n+i_j}) + c_{0m}}$$

and demand that it define sequences which approach the number $\beta^{1/q}$ with an order of rate of convergence k, where k is an integer to be determined, β is a rational number, and $\beta^{j/q}$ is irrational ($j = 1, \cdots, q-1$). That is, set $S_{n+j} = \beta + \theta_j\epsilon^{k^j}$ ($j = 0, 1, \cdots, m-1$), expand the right member of (33) in powers of ϵ and demand that the constant term equal $\beta^{1/q}$, and that the coefficients of ϵ^t vanish for $t = 1, 2, \cdots, k^m - 1$. These demands impose k^m linear homogeneous relations upon the coefficients of the given equation (33); as can readily be seen by dividing the numerator by the denominator. The k^m linear relations may contain some of the θ_i ($i = 0, \cdots, m-1$) and some fractional powers of β. In order to insure that the coefficients of the difference equation (33) be rational, the coefficients of θ_i and the fractional power of β in these linear relations must vanish.

The maximum value for k is determined by the number of the relations that the coefficients of (33) may satisfy, that is, by the number of coefficients. This number[8] is

$$\sum_{j=1}^{q}\frac{m(m+1)\cdots(m+j-2)(m+2j-1)}{j!} + 1.$$

Since, when $m = 1$ it is $\sum_{j=1}^{q}[2] + 1 = 2q + 1$, the coefficients of a first order difference equation may be chosen so as to satisfy at least $2q$ linear homogeneous relations. Therefore k may be at least 2, for when $k = 2$ the coefficients are only restricted by $2q$ relations, q relations which insure that the constant term in the power series in ϵ equal $\beta^{1/q}$ and q relations which insure that the coefficient of ϵ equal zero. Hence if a difference equation of higher than first order defines sequences whose order of rate of convergence, k, is greater than that of sequences defined by a first order difference equation, k must be at least 3.

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[8] In order to see this, write

$$\sum_{i_1=0}^{m-1}\cdots\sum_{i_j\geqq i_{j-1}}^{m} = \sum_{i_1=0}^{m-1}\cdots\sum_{i_j\geqq i_{j-1}}^{m-1} + \sum_{i_1=0}^{m-1}\cdots\sum_{i_{j-1}\geqq i_{j-2}}^{m-1}$$

The number of terms in the first sum on the right is $m(m+1)\cdots(m+j-1)/j!$. Therefore, the total number of terms of j^{th} degree is

$$\frac{m(m+1)\cdots(m+j-1)}{j!} + \frac{m(m+1)\cdots(m+j-2)}{(j-1)!} = \frac{m(m+1)\cdots(m+j-2)(m+2j-1)}{j!}.$$

Now consider difference equation of the second order. When $m = 2$ the number of terms in (33) is

$$\sum_{j=1}^{q} (2j + 1) + 1 = \frac{2q(q + 1)}{2} + q + 1 = (q + 1)^2.$$

If a sequence converges to $\beta^{1/q}$ with an order of rate of convergence 3, the coefficients must satisfy $18q$ relations—q relations to make the first term of the series in ϵ equal $\beta^{1/q}$, $2q$ to make the coefficients of ϵ and ϵ^2 vanish; $6q$ to make the coefficients of ϵ^3, ϵ^4, ϵ^5 vanish (twice as many here since θ_1 may be irrational and may appear in each of these relations); and $9q$ to make the coefficients of ϵ^6, ϵ^7 and ϵ^8 vanish (θ_1 and θ_1^2 may enter in each of these three relations). Since there are $(q + 1)^2$ coefficients in the difference equation, if $q \geqq 17$ these $18q$ relations may be satisfied, for then the number of homogeneous linear equations is less than the number of undetermined coefficients. Hence, for $q \geqq 17$ there exists a second order difference equation which defines sequences that approach $\beta^{1/q}$ with an order of rate of convergence 3.

Now if we can prove that no first order difference equation of 17th degree may define sequences which approach $\beta^{1/17}$ with an order of rate of convergence 3 our proof is complete. It follows from Theorem 7, that there is no loss in generality if we only consider equations with constant coefficients. The general first order difference equation of 17th degree is

$$S_{n+1} = -\frac{\sum_{j=0}^{17} a_j S_n^j}{\sum_{j=0}^{16} b_j S_n^j}.$$

If this equation defines sequences which converge to $\beta^{1/17}$ with an order of convergence 3, when we set $S_n = \beta^{1/17} + \epsilon$ and expand the right member in powers of ϵ, the constant term will be equal to $\beta^{1/17}$ and the coefficients of ϵ and ϵ^2 equal to zero, viz.

$$a_0 + (a_1 + b_0)\beta^{1/17} + (a_2 + b_1)\beta^{2/17} + \cdots + (a_{17} + b_{16})\beta = 0$$

$$a_1 + (2a_2 + b_1)\beta^{1/17} + (3a_3 + 2b_2)\beta^{2/17} + \cdots + (17a_{17} + 16b_{16})\beta^{16/17} = 0$$

$$a_2 + (3a_3 + b_2)\beta^{1/17} + (6a_4 + 3b_3)\beta^{2/17} + \cdots + (136a_{17} + 120b_{16})\beta^{15/17} = 0$$

If the a's and b's are to have rational values they must satisfy the following equations

$$a_1 + b_0 = 0, \quad a_2 + b_1 = 0, \quad \cdots, a_{16} + b_{15} = 0, \quad a_0 + (a_{17} + b_{16})\beta = 0$$

$$a_1 = 0, \quad 2a_2 + b_1 = 0, \quad 3a_3 + 2b_1 = 0, \cdots, \quad 17a_{17} + 16b_{16} = 0$$

$$a_2 = 0, \quad 3a_3 + b_2 = 0, \quad 6a_4 + 3b_3 = 0, \cdots, \quad 136a_{17} + 120b_{16} = 0,$$

which have the unique solution $a_i = 0$, $b_j = 0$ $(i = 0, 1, \cdots, 17)$ $(j = 0, 1, \cdots, 16)$. Therefore no such equation exists. Q.E.D.

Next we show that the sequences defined by a first order difference equation may converge to an algebraic number more rapidly than those defined by all other difference equation of higher order and of the same degree.

THEOREM 9. *The difference equation of second degree that defines sequences which converge most rapidly to the irrational number $\beta^{\frac{1}{2}}$ where β is rational, is the one of first order.*

PROOF. Consider the general second degree difference equation of the m^{th} order

$$\sum_{r=0}^{l}\left[\sum_{j=0}^{m-1}\sum_{i\geq j}^{m} c_{rij}n^{-r}S_{n+j}S_{n+i} + \sum_{i=0}^{m} c_{ri}n^{-r}S_{n+i} + n^{-r}c_r\right] = 0. \tag{34}$$

When it defines sequences that approach $\beta^{\frac{1}{2}}$,

$$\sum_{j=0}^{m-1}\sum_{i\geq j}^{m} c_{0ij}\beta + \sum_{i=0}^{m} c_{0i}\beta^{\frac{1}{2}} + c_0 = 0. \tag{35}$$

and the transformed equation is

$$\begin{aligned}\sum_{r=0}^{l} n^{-r}&\left[\sum_{j=0}^{m-1}\sum_{i\geq j}^{m} c_{rij}(\bar{S}_{n+j}\bar{S}_{n+i} + \beta^{\frac{1}{2}}\bar{S}_{n+j} + \beta^{\frac{1}{2}}\bar{S}_{n+i}) + \sum_{i=0}^{m} c_{ri}\bar{S}_{n+i}\right]\\ &+ \sum_{r=1}^{l} n^{-r}\left[\sum_{j=0}^{m-1}\sum_{i\geq j}^{m} c_{rij}\beta + \sum_{i=0}^{m} c_{ri}\beta^{\frac{1}{2}} + c_r\right] = 0\end{aligned} \tag{36}$$

If a difference equation of the form (36) defines sequences which converge to zero with the maximum rate, then, according to Theorem 5

$$\sum_{j=0}^{m-1}\sum_{i\geq j}^{m} c_{rij}\beta + \sum_{i=0}^{m} c_{ri}\beta^{\frac{1}{2}} + c_r = 0 \qquad (r = 1, 2, \cdots, l) \tag{35'}$$

and

$$\sum_{i\geq j}^{m} \epsilon_{ij}c_{rji}\beta^{\frac{1}{2}} + c_{rj} = 0 \qquad (j = 0, 1, \cdots, m-1)(r = 0, 1, \cdots, l) \tag{37}$$

where

$$\epsilon_{ij} = \begin{cases}1 & i \neq j\\ 2 & i = j.\end{cases}$$

Therefore when $\beta^{\frac{1}{2}}$ is an irrational number and the coefficients of the difference equation are rational numbers, $c_{rj} = 0$, $(j = 0, 1, \cdots m - 1)$ $(r = 0, 1, \cdots, l)$ And since from (35) and (35′) $\sum_{i=0}^{m} c_{ri} = 0$ $(r = 0, 1, \cdots, l)$, $c_{rm} = 0$ $(r = 0, 1, \cdots, l)$.

Now assume our theorem is false, then there is a difference equation of the form (34) of higher than first order that defines sequences which approach $\beta^{\frac{1}{2}}$ with an order of convergence 2. Writing (36) in the form

$$\bar{S}_{n+m} = -\frac{\sum_{r=0}^{l} \frac{1}{n^r} \sum_{j=0}^{m-1} \sum_{i \geqq j}^{m-1} c_{rij}(\bar{S}_{n+j}\bar{S}_{n+i} + \beta^{\frac{1}{2}}\bar{S}_{n+j}, + \beta^{\frac{1}{2}}\bar{S}_{n+i})}{\sum_{r=0}^{l} \frac{1}{n^r} \sum_{j=0}^{m-1} c_{rmj}(\bar{S}_{n+j} + \beta^{\frac{1}{2}})},$$

we can see that if $\bar{S}_{n+m} = \theta_m \epsilon^{2^m}$ when $S_{n+i} = \theta_i \epsilon^{2^i}$ $(i = 0, 1, \cdots, m-1)$

$$c_{rij} = 0 \qquad (i, j = 0, 1, \cdots, m-2)(r = 0, \cdots, l)$$

$$c_{r,m-2,m-1} = 0 \qquad (r = 0, 1, 2, \cdots, l).$$

These results combined with the equation (37) and (35) give

$$c_{rmj} = 0 \quad (j = 0, 1, \cdots, m-2)(r = 0, 1, \cdots, l)$$

and

$$(c_{r,m-1,m} + c_{r,m-1,m-1})\beta + c_r = 0 \qquad (r = 0, 1, \cdots, l)$$

$$2c_{r,m-1,m-1} + c_{r,m-1,m} = 0 \qquad (r = 0, 1, \cdots, l).$$

Therefore the unique equation of the form (34) which defines sequences that approach $\beta^{\frac{1}{2}}$ with an order of convergence 2 is

$$2S_{n+m}S_{n+m-1} - S_{n+m-1}^2 - \beta = 0$$

but this equation is equivalent to the first order difference equation

$$2S_{n+1}S_n - S_n^2 - \beta = 0 \qquad \text{Q.E.D.}$$

In summarizing we can only say that, in general, there is no direct relation between the order of a difference equation and the order of rate of convergence of its sequences.

4. Minimum degree for a first order difference equation whose sequences approach $\beta^{1/p}$ with a given order of convergence.

THEOREM 10. *There exists a first order difference equation of q^{th} degree, $q = kp$, that define sequences which converge to $\beta^{1/p}$, where β is rational and $\beta^{j/p}$ is irrational $(j = 1, 2, \cdots, p-1)$, with an order of rate of convergence $2k$, and no first order difference equation of degree kp may define sequences which converge to $\beta^{1/p}$ with an order of convergence greater than $2k$.*

PROOF. The transformed equation of the first order difference equation of q^{th} degree,

$$S_{n+1} = \frac{-\sum_{j=0}^{q} a_j S_n^j}{\sum_{j=0}^{q-1} b_j S_n^j} \tag{38}$$

is

$$\bar{S}_{n+1} = -\frac{\sum_{j=0}^{q} a_j(\bar{S}_n - \beta^{1/p})^j}{\sum_{j=0}^{q-1} b_j(\bar{S}_n - \beta^{1/p})^j} - \beta^{1/p}. \tag{39}$$

If the coefficients a_j, b_i $(j = 0, 1, \cdots, m)$ $(i = 0, 1, \cdots, m - 1)$ may be determined so that the coefficients of $\bar{S}_n^t$ in the numerator of the left member of (39) vanish $(t = 0, 1, \cdots, r)$, then (38) defines sequences which approach $\beta^{1/p}$ with an order of convergence $r + 1$. Setting the coefficients of $\bar{S}_n^j$ $(t = 0, 1, \cdots, r)$ equal to zero we obtain the following linear homogeneous equation in the a's and b's:

$$\begin{aligned}
&\sum_{j=0}^{q-1} (a_j\beta^{j/p} + b_j\beta^{(j+1)/p}) + a_q\beta^{q/p} = 0 \\
&\sum_{j=0}^{q-1} j(a_j\beta^{(j-1)/p} + b_j\beta^{j/p}) + qa_q\beta^{(q-1)/p} = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\sum_{j=r}^{q-1} \binom{j}{r} (a_j\beta^{(j-r)/p} + b_j\beta^{(j-r+1)/p}) + \binom{q}{r}\beta^{(q-p)/p}a_q = 0.
\end{aligned} \tag{40}$$

Since by hypothesis $\beta^{j/p}$ is irrational $(j = 1, 2, \cdots, p - 1)$, and since the a's and b's are to be rational each of the equation (40) divide into p equations. Therefore, the demand that the sequences approach $\beta^{1/p}$ with an order of convergence $2k$ imposes $2kp$ linear homogeneous relations upon the $2kp + 1$ coefficients of (38). Hence there is always a solution.

Assume that the second part of the theorem is false, then the a's and b's will satisfy the relations (40) when $r = 2k$, that is to say, they will satisfy $(2k + 1)p$ linear homogeneous equations. These $(2k + 1)p$ equations divide into p systems of equation. One with $2k + 1$ unknowns,

$$\left\{\begin{aligned}
&a_0 + \beta b_{p-1} + \beta a_p + \beta^2 b_{2p-1} + \beta^2 a_{2p} + \cdots + \beta^k b_{kp-1} + \beta^k a_q = 0 \\
&\binom{p-1}{1}\beta b_{p-1} + \binom{p}{1}\beta a_p + \binom{2p+1}{1}\beta^2 b_{2p-1} + \binom{2p}{1}\beta^2 a_{2p} + \cdots \\
&\qquad\qquad + \binom{kp-1}{1}\beta^k b_{kp-1} + \binom{q}{1}\beta^k a_q = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\binom{p-1}{2k}\beta b_{p-1} + \binom{p}{2k}\beta a_p + \binom{2p+1}{2k}\beta^2 b_{2p-1} + \binom{2p}{2k}\beta^2 a_{2p} + \cdots \\
&\qquad\qquad + \binom{kp-1}{2k}\beta^k b_{kp-1} + \binom{q}{2k}\beta^k a_q = 0
\end{aligned}\right. \tag{41}$$

$$\bar{S}_{n+m} = -\frac{\sum_{r=0}^{l} \frac{1}{n^r} \sum_{j=0}^{m-1} \sum_{i \geqq j}^{m-1} c_{rij}(\bar{S}_{n+j}\bar{S}_{n+i} + \beta^{\frac{1}{2}}\bar{S}_{n+j}, + \beta^{\frac{1}{2}}\bar{S}_{n+i})}{\sum_{r=0}^{l} \frac{1}{n^r} \sum_{j=0}^{m-1} c_{rmj}(\bar{S}_{n+j} + \beta^{\frac{1}{2}})},$$

we can see that if $\bar{S}_{n+m} = \theta_m \epsilon^{2^m}$ when $S_{n+i} = \theta_i \epsilon^{2^i}$ $(i = 0, 1, \cdots, m-1)$

$$c_{rij} = 0 \qquad (i, j = 0, 1, \cdots, m-2)(r = 0, \cdots, l)$$

$$c_{r,m-2,m-1} = 0 \qquad (r = 0, 1, 2, \cdots, l).$$

These results combined with the equation (37) and (35) give

$$c_{rmj} = 0 \quad (j = 0, 1, \cdots, m-2)(r = 0, 1, \cdots, l)$$

and

$$(c_{r,m-1,m} + c_{r,m-1,m-1})\beta + c_r = 0 \qquad (r = 0, 1, \cdots, l)$$

$$2c_{r,m-1,m-1} + c_{r,m-1,m} = 0 \qquad (r = 0, 1, \cdots, l).$$

Therefore the unique equation of the form (34) which defines sequences that approach $\beta^{\frac{1}{2}}$ with an order of convergence 2 is

$$2S_{n+m}S_{n+m-1} - S_{n+m-1}^2 - \beta = 0$$

but this equation is equivalent to the first order difference equation

$$2S_{n+1}S_n - S_n^2 - \beta = 0 \qquad \text{Q.E.D.}$$

In summarizing we can only say that, in general, there is no direct relation between the order of a difference equation and the order of rate of convergence of its sequences.

4. Minimum degree for a first order difference equation whose sequences approach $\beta^{1/p}$ with a given order of convergence.

THEOREM 10. *There exists a first order difference equation of q^{th} degree, $q = kp$, that define sequences which converge to $\beta^{1/p}$, where β is rational and $\beta^{j/p}$ is irrational $(j = 1, 2, \cdots, p-1)$, with an order of rate of convergence $2k$, and no first order difference equation of degree kp may define sequences which converge to $\beta^{1/p}$ with an order of convergence greater than $2k$.*

PROOF. The transformed equation of the first order difference equation of q^{th} degree,

$$S_{n+1} = \frac{-\sum_{j=0}^{q} a_j S_n^j}{\sum_{j=0}^{q-1} b_j S_n^j} \tag{38}$$

is

$$\bar{S}_{n+1} = -\frac{\sum_{j=0}^{q} a_j(\bar{S}_n - \beta^{1/p})^j}{\sum_{j=0}^{q-1} b_j(\bar{S}_n - \beta^{1/p})^j} - \beta^{1/p}. \tag{39}$$

If the coefficients a_j, b_i $(j = 0, 1, \cdots, m)$ $(i = 0, 1, \cdots, m - 1)$ may be determined so that the coefficients of $\bar{S}_n^t$ in the numerator of the left member of (39) vanish $(t = 0, 1, \cdots, r)$, then (38) defines sequences which approach $\beta^{1/p}$ with an order of convergence $r + 1$. Setting the coefficients of $\bar{S}_n^i$ $(t = 0, 1, \cdots, r)$ equal to zero we obtain the following linear homogeneous equation in the a's and b's:

$$\begin{aligned}
&\sum_{j=0}^{q-1} (a_j\beta^{j/p} + b_j\beta^{(j+1)/p}) + a_q\beta^{q/p} = 0\\
&\sum_{j=0}^{q-1} j(a_j\beta^{(j-1)/p} + b_j\beta^{j/p}) + qa_q\beta^{(q-1)/p} = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sum_{j=r}^{q-1} \binom{j}{r} (a_j\beta^{(j-r)/p} + b_j\beta^{(j-r+1)/p}) + \binom{q}{r} \beta^{(q-p)/p} a_q = 0.
\end{aligned} \tag{40}$$

Since by hypothesis $\beta^{j/p}$ is irrational $(j = 1, 2, \cdots, p - 1)$, and since the a's and b's are to be rational each of the equation (40) divide into p equations. Therefore, the demand that the sequences approach $\beta^{1/p}$ with an order of convergence $2k$ imposes $2kp$ linear homogeneous relations upon the $2kp + 1$ coefficients of (38). Hence there is always a solution.

Assume that the second part of the theorem is false, then the a's and b's will satisfy the relations (40) when $r = 2k$, that is to say, they will satisfy $(2k + 1)p$ linear homogeneous equations. These $(2k + 1)p$ equations divide into p systems of equation. One with $2k + 1$ unknowns,

$$\left\{\begin{aligned}
&a_0 + \beta b_{p-1} + \beta a_p + \beta^2 b_{2p-1} + \beta^2 a_{2p} + \cdots + \beta^k b_{kp-1} + \beta^k a_q = 0\\
&\binom{p-1}{1}\beta b_{p-1} + \binom{p}{1}\beta a_p + \binom{2p+1}{1}\beta^2 b_{2p-1} + \binom{2p}{1}\beta^2 a_{2p} + \cdots\\
&\qquad\qquad + \binom{kp-1}{1}\beta^k b_{kp-1} + \binom{q}{1}\beta^k a_q = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\binom{p-1}{2k}\beta b_{p-1} + \binom{p}{2k}\beta a_p + \binom{2p+1}{2k}\beta^2 b_{2p-1} + \binom{2p}{2k}\beta^2 a_{2p} + \cdots\\
&\qquad\qquad + \binom{kp-1}{2k}\beta^k b_{kp-1} + \binom{q}{2k}\beta^k a_q = 0
\end{aligned}\right. \tag{41}$$

and $(p - 1)$ of the form

$$
(42)\quad \left\{
\begin{aligned}
& b_j + a_{j+1} + \beta b_{j+p} + \beta a_{j+p+1} + \cdots + \beta^{k-1} b_{j+(k-1)p} + \beta^k a_{j+(k-1)p+1} = 0 \\
& \binom{j}{1} b_j + \binom{j+1}{1} a_{j+1} + \binom{j+p}{1} \beta b_{j+p} + \binom{j+p+1}{1} \beta a_{j+p+1} \\
& \qquad + \cdots + \binom{j+(k-1)p}{1} \beta^{k-1} b_{j+(k-1)p} \\
& \qquad\qquad + \binom{j+(k-1)p+1}{1} \beta^k a_{j+(k-1)p+1} = 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& \binom{j}{2k} b_j + \binom{j+1}{2k} a_{j+1} + \binom{j+p}{2k} \beta b_{j+p} + \binom{j+p+1}{2k} \beta a_{j+p+1} \\
& \qquad + \cdots + \binom{j+(k-1)p}{2k} \beta^{k-1} b_{j+(k-1)p} \\
& \qquad\qquad + \binom{j+(k-1)p+1}{2k} \beta^k a_{j+(k-1)p+1} = 0
\end{aligned}
\right.
$$

$(j = 0, 1, 2, \cdots, p - 2)$. When $j = 0$ the last equation of the system may not appear but in all cases there will be at least $2k$ equations in each of the last $p - 1$ systems. Therefore, since all of the systems of equations have at least as many equations as unknowns there are no non-trivial solutions unless a determinant of one of the systems is zero. The determinants of the coefficients are all of the form

$$
\begin{vmatrix}
\binom{j}{0} & \binom{j+1}{0} & \binom{j+p}{0} & \binom{j+p+1}{0} & \cdots & \binom{j+mp}{0} & \binom{j+mp+1}{0} \\
\binom{j}{1} & \binom{j+1}{1} & \binom{j+p}{1} & \binom{j+p+1}{1} & \cdots & \binom{j+mp}{1} & \binom{j+mp+1}{1} \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\binom{j}{r} & \binom{j+1}{r} & \binom{j+p}{r} & \binom{j+p+1}{r} & \cdots & \binom{j+mp}{r} & \binom{j+mp+1}{r}
\end{vmatrix}.
$$

Such a determinant is equal to

$$
\frac{1}{2!3!\cdots r!}
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
j & j+1 & j+p & \cdots & j+mp+1 \\
j^2 & (j+1)^2 & (j+p)^2 & \cdots & (j+mp+1)^2 \\
\cdot & \cdot & \cdot & \cdots & \cdot \\
j^r & (j+1)^r & (j+p)^r & \cdots & (j+mp+1)^r
\end{vmatrix},
$$

which is different from zero, for it is a Vandermonde determinant. Therefore, we have a contradiction. Q.E.D.

It is not difficult to see that the equation of q^{th} degree, $q = kp$, which defines sequences that converge to $\beta^{1/p}$ with an order of convergence $2k$ is unique. The solutions of the first $2k$ equations of the last $p - 1$ systems (42) are all zero and the solutions of the first $2k$ equations are of the form $c_i k$. Hence, when the factor k is removed from the numerator and the denominator, we obtain the equation.

THEOREM 11. *There is a first order difference equation of degree $kp + 1$ that defines sequences which converge to $\beta^{1/p}$, where β is rational and $\beta^{j/p}$ is irrational $(j = 1, 2, \cdots, p - 1)$, more rapidly than those defined by any first order difference equation of degree q, where $(k + 1)p > q > kp + 1$.*

PROOF. Consider the difference equation (38) for $q = (k + 1)p - 1$. First, assume that the sequences defined by this equation approach $\beta^{1/p}$ with an order of convergence $2k + 2$. Then the coefficients satisfy $2k + 2$ relations (40). These $2k + 2$ relations give $(2k + 2)p$ homogeneous linear equations in the a's and b's. And the linear equations divide into p systems of $2k + 2$ equations each. One of the systems contains $2k + 1$ unknowns and the other $p - 1$ systems each contain $2k + 2$ unknowns. None of these systems have non-trivial solutions for the determinants of the coefficients are composed of columns of binomial coefficients which we proved, in the proof of Theorem 10 to be non-vanishing. Hence our assumption is false.

Second, assume that the sequences have an order of rate of convergence $2k + 1$, then the a's and the b's must satisfy $2k + 1$ relations (40). These $2k + 1$ relations give p systems of homogeneous linear equations (41) and (42). One system contains $2k + 1$ unknowns and the other $p - 1$ systems contain $2k + 2$ unknowns each. The solutions of the first system must all be zero, while each of the other $p - 1$ systems may be solved for $2k + 1$ of their unknowns in terms of one of the others, for again, all determinants of the coefficients are different from zero.

We construct a matrix of $2(k + 1)$ columns and $(2k + 1)(p - 1)$ rows, from the coefficients of the $p - 1$ systems of equations, where the coefficients of $a_i\beta^j$, $a_{i+1}\beta^j, \cdots, a_{i+p-1}\beta^j$ lie in the same column $(i = 1, p + 1, \cdots, kp + 1)$ and the coefficients $b_l\beta^j, b_{l+1}\beta^j, \cdots, b_{l+p-1}\beta^j$ lie in the same column $(l = 0, \text{p}, 2p, \cdots, kp)$. That is, the matrix

$$\left\| \begin{matrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
0 & \binom{1}{1} & \binom{p}{1} & \binom{p+1}{1} & \cdots & \binom{kp}{1} & \binom{kp+1}{1} \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & \binom{p}{2k} & \binom{p+1}{2k} & \cdots & \binom{kp}{2k} & \binom{kp+1}{2k} \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\binom{1}{1} & \binom{2}{1} & \binom{p+1}{1} & \binom{p+2}{1} & \cdots & \binom{kp+1}{1} & \binom{kp+2}{1} \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & \binom{2}{2k} & \binom{p+1}{2k} & \binom{p+2}{2k} & \cdots & \binom{kp+1}{2k} & \binom{kp+2}{2k} \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\binom{p-2}{1} & \binom{p-1}{1} & \binom{2p-2}{1} & \binom{2p-1}{1} & \cdots & \binom{(k+1)p-2}{1} & \binom{(k+1)p-1}{1} \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\binom{p-2}{2k} & \binom{p-1}{2k} & \binom{2p-2}{2k} & \binom{2p-1}{2k} & \cdots & \binom{(k+1)p-2}{2k} & \binom{(k+1)p-1}{2k}
\end{matrix} \right\| .$$

Since the binomial coefficients obey the law

$$\binom{r}{n-1} + \binom{r}{n} = \binom{r+1}{n}$$

any one of the rows of this matrix is linearly dependent upon the first $2k + 1$ rows. Hence the rank of the matrix is $2k + 1$.

This tells us that if we solve the first system of equations for $2k + 1$ of the unknowns in terms of a_i that the second set of equations has the same solutions where the subscripts are increased by one, and so on. If we solve $p - 1$ systems of equations in terms of $a_{kp+1}, a_{kp+2}, \cdots, a_{(k+1)p-1}$ respectively, then the given equation (38) takes the form

$$S_{n+1} = \frac{\phi_1(S_n) \sum_{j=0}^{p-1} a_{kp+j} S_n^{j-1}}{\phi_2(S_n) \sum_{j=1}^{p-1} a_{kp+j} S_n^{j-1}}$$

or

$$S_{n+1} = \frac{\phi_1(S_n)}{\phi_2(S_n)}, \tag{43}$$

where $\phi_1(S_n)$ and $\phi_2(S_n)$ are of degrees $kp + 1$ and kp respectively. Since the equation (43) is unique, our proof is complete. Q.E.D.

The two previous theorems have dealt only with difference equations with constant coefficients. In the following theorem we assume that the coefficients are polynomials in n.

THEOREM 12. *There exists some first order difference equations with constant coefficients that define rational sequences which converge to* $\beta^{1/p}$, *where* $\beta^{j/p}$ *is irrational* ($j = 1, 2, \cdots, p - 1$) *more rapidly than the sequences defined by any first order difference equation whose coefficients are polynomials in* n.

PROOF. If the difference equation

$$S_{n+1} = -\frac{\sum_{i=0}^{l} n^i \sum_{j=0}^{q} a_{ij} S_n^j}{\sum_{i=0}^{l} n^i \sum_{j=0}^{q-1} b_{ij} S_n^j} \tag{44}$$

defines sequences which converge to $\beta^{1/p}$ with an order of convergence λ, then

$$\begin{aligned}
&\sum_{j=0}^{q-1} (a_{ij}\beta^{j/p} + b_{ij}\beta^{(j+1)/p}) + a_{iq}\beta^{q/p} = 0 && (i = 0, 1, \cdots, l) \\
&\sum_{j=0}^{q-1} j(a_{ij}\beta^{(j-1)/p} + b_{ij}\beta^{j/p}) + q a_{iq}\beta^{(q-1)/p} = 0 && (i = 0, 1, \cdots, l) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\sum_{j=0}^{q-1} \binom{j}{r-1} [a_{ij}\beta^{(j-r+1)/p} + b_{ij}\beta^{(j-r)/p}] \\
&\qquad + \binom{q}{r-1} a_{iq}\beta^{(q-r+1)/p} = 0, && (i = 0, 1, \cdots, l).
\end{aligned} \tag{45}$$

When $l = 0$ we know from the previous theorems that there is a maximum order λ for the rate of convergence for the sequences. That is, the number of relations for the determination of a_{0j} and b_{0j} cannot exceed λ. The relations (45) are similar for all values of i. Hence when $l > 0$, the number of such relations cannot exceed λ, without all a_{ij} and b_{ij} vanishing.

When $q = kp$ we know, after Theorem 10 that $\lambda = 2k$. Each of these $2k$ relations (45) gives, for each i, q linear homogeneous equations on the coefficients. Since the relations are the same for all i, when the last system of equations are solved in terms of a_{ir} $(i = 0, 1, \cdots, l)$ (45) reduces to

$$S_{n+1} = \frac{\phi_1(S_n) \sum_{i=0}^{l} n^i a_{ir}}{\phi_2(S_n) \sum_{i=0}^{l} n^i a_{ir}}$$

or

$$S_{n+1} = \frac{\phi_1(S_n)}{\phi_2(S_n)},$$

where ϕ_1 and ϕ_2 are polynomials is S_n with constant coefficients.

When $q = (k + 1)p - 1$, then as in the proof of Theorem 11 for $l = 0$

$$S_{n+1} = \frac{\phi_3(S_n) \sum_{j=0}^{p-1} a_{0kp+j} S_n^{j-1}}{\phi_4(S_n) \sum_{j=0}^{p-1} a_{0kp+j} S_n^{j-1}}$$

and since the relations (45) are similar for all i

$$S_{n+1} = \frac{\phi_3(S_n) \sum_{i=0}^{l} \sum_{j=0}^{p-1} a_{ikp+j} S_n^{j-1}}{\phi_4(S_n) \sum_{i=0}^{l} \sum_{j=0}^{p-1} a_{ikp+j} S_n^{j-1}},$$

or

$$S_{n+1} = \frac{\phi_3(S_n)}{\phi_4(S_n)},$$

where ϕ_3 and ϕ_4 are polynomials with constant coefficients. Q.E.D.

EXAMPLE: When $k = 1$, the first order difference equation of p^{th} degree whose sequences converge most rapidly to $\beta^{1/p}$, where β is rational and $\beta^{j/p}$ is irrational $(j = 1, 2, \cdots, p - 1)$, is

$$pS_{n+1}S_n^{p-1} - (p - 1)S_n^p - \beta = 0 \tag{46}$$

and the one of $(p + 1)^{\text{st}}$ degree is

$$(p + 1)S_{n+1}S_n^p - (p - 1)S_n^{p+1} + (p - 1)\beta S_{n+1} - (p + 1)S_n\beta = 0. \tag{47}$$

The sequences defined by (46) have an order of convergence 2 and those defined by (47) have an order of convergence 3.

It is interesting to note that equation (46) is *Newton's algorithm for the approximation of* $\beta^{1/p}$.

5. Approximation of $\beta^{i/p}$. In practical work it is often valuable to know an upper bound for the error of a given term of a sequence, i.e., an upper bound for the difference between a term and the limit of the sequence, or in other words, if the terms of the sequence are written in decimal form it is important to know a means of asserting that a given term of the sequence is correct to at least R decimal places.

THEOREM 13. *If a sequence approaches the real number* $\beta^{1/p}$, *such that when* $| S_n - \beta^{1/p} | = \epsilon$ *then* $| S_{n+1} - \beta^{1/p} | \leqq C\epsilon^k$, *where C is a constant independent of* ϵ, (i.e. the sequence has an order of convergence k) *and if,* ϵ *is so small that* $2C\epsilon^k < \epsilon$, *and if the first i decimal places of the two consecutive terms of the sequence are the same,* $(i > 0)$; *then, the second of the terms cannot differ from* $\beta^{1/p}$ *by as much as* $2^k(10)^{-ki}C$. *And when* $C < (10)^j$ *the second of the terms is correct to* $ki - j - k/3$ *decimal places.*

PROOF. Let ϵ_n denote the error of S_n. Since $C\epsilon_n^{k-1} < \frac{1}{2}$,

$$\begin{aligned}\epsilon_n &< 2\epsilon_n \,| 1 - C\epsilon_n^{k-1} | \\ &= 2 \,| \epsilon_n - C\epsilon_n^k | \leqq 2 \,|\, | S_n - \beta^{1/p} | - | S_{n+1} - \beta^{1/p} | \,| \leqq 2 \,| S_n - S_{n+1} |.\end{aligned}$$

Now

$$| S_n - S_{n+1} | < (10)^{-i},$$

hence

$$\epsilon_{n+1} \leqq C\epsilon_n^k < C2^k(10)^{-ki}.$$

And when

$$C < 10^j$$

$$\epsilon_{n+1} < 2^k(10)^{j-ki} < 10^{k(-i+\frac{1}{3})+j}.$$

Therefore, S_{n+1} is correct to at least $ki - j - k/3$ decimal places.

EXAMPLE: Consider the sequences defined by (27). For this equation $k = 2$, $| C | < 1$ therefore $j = 0$. For the sequence with initial condition $S_0 = 1$, S_3 and S_4 differ only in the sixth decimal place, hence S_4 is correct to 9 decimal places. By the same method of calculation S_5 is correct to 21 decimal places.

Although it is possible to set up a difference equation whose sequences converge as rapidly as one desired, there seems to be less calculation in the approximation of an irrational number $\beta^{1/p}$ if the sequences are defined by the simple difference equation (46) *Newton's algorithm*, which may be written in the form

$$S_{n+1} = \frac{p-1}{p} S_n + \frac{\beta}{pS_n^{p-1}}. \tag{48}$$

We know that the order of rate of convergence of its sequences is 2. Now let us determine its region of strong convergence.

THEOREM 14. *If* $\beta^{1/p} > 0$, *the sequences defined by* (48) *converge to* $\beta^{1/p}$ *for all positive initial values* S_0. *Moreover, they approach* $\beta^{1/p}$ *monotonically when the initial values are* $> \beta^{1/p}$.

PROOF. The transformation equation is

$$p(\bar{S}_{n+1} + \beta^{1/p})(\bar{S}_n + \beta^{1/p})^{p-1} - (p - 1(\bar{S}_n + \beta^{1/p})^p - \beta = 0$$

or

$$\bar{S}_{n+1} = \bar{S}_n\left[1 - \frac{1}{p} - \frac{\beta^{1/p}(\bar{S}_n + \beta^{1/p})^{p-1} - \beta}{p\bar{S}_n(\bar{S}_n + \beta^{1/p})^{p-1}}\right].$$

For positive values of $\bar{S}_n$ the expression in the brackets is > 0 and $< 1 - 1/p$. Therefore, the sequences S_n decrease monotonically to zero for initial values $\bar{S}_0 > 0$. This is equivalent to the statement that all sequences $\{S_n\}$ of (48) determined by initial conditions $> \beta^{1/p}$ converge monotonically to $\beta^{1/p}$. Hence the second part of the theorem is true.

If $0 < S_n < \beta^{1/p}$, then $S_{n+1} - S_n = (1/p)\{-S_n + \beta S_n^{1-p}\} > 0$, so that $S_{n+1} > S_n$. Hence, either the sequence $\{S_n\}$ is monotone increasing tending to a limit $\leqq \beta^{1/p}$, or some term of the sequence is greater than $\beta^{1/p}$ and from this term on the sequence decreases monotonically to $\beta^{1/p}$. In the first case the sequence cannot approach a limit K, $0 < K < \beta^{1/p}$ for the only finite limits for the sequence are $\beta^{1/p}$ and $-\beta^{1/p}$ since the limit of the difference equation is $S_0^p = \beta$. Hence our proof is complete. Q.E.D.

Since irrational numbers of the form $\beta^{1/p}$ are usually accurately computed by means of a binomial series, it is interesting to compare the rate of convergence of the sequence of partial sums of a binomial series with that of a sequence defined by a difference equation.

THEOREM 15. *A sequence of partial sums of a binomial series cannot converge to* $\beta^{1/p}$ *with a order of convergence* 2.

PROOF. CASE I. Positive term series, $(1 - 1/a)^l = 1 + b_1 + b_2 + \cdots + b_n + \cdots$. Assume that the theorem is false, then the sequence of partial sums has an order of rate of convergence 2; i.e. if

$$\beta^{1/p} - S_n \mp \epsilon_n \quad \text{and} \quad \beta^{1/p} - S_{n+1} = \epsilon_{n+1} \quad \text{then} \quad \epsilon_{n+1} < C\epsilon_n^2,$$

where C is a constant independent of ϵ_n. For a rapidly convergent series $b_{n+1}/b_n < r < 1$ for $n > n_0$. The error ϵ_n of the partial sum S_n is less than $b_n\left(\frac{r}{1-r}\right)$ and is greater than b_{n+1}. Therefore

$$\epsilon_n < b_n\left(\frac{r}{1-r}\right) = \binom{l}{n}\frac{1}{(-a)^n}\cdot\left(\frac{r}{1-r}\right),$$

$$\binom{l}{n+1}\left(\frac{1}{-a}\right)^{n+1} = b_{n+1} < \epsilon_{n+1}.$$

Therefore, if the theorem is false,

$$\binom{l}{n+1}\left(\frac{1}{-a}\right)^{n+1} < C\binom{l}{n}^2\frac{1}{a^{2n}}\left(\frac{r}{1-r}\right)^2$$

or

$$\left|\frac{(l-n)}{(n+1)}\frac{1}{a}\right| < C\left|\frac{l(l-1)\cdots(l-n+1)}{n!}\frac{1}{a^n}\right|\left(\frac{r}{1-r}\right)^2.$$

This last inequality cannot hold for all n, if $a > 1$. But, since the series is convergent, $a > 1$. Hence we have a contradiction.

Case II. Alternating series. $(1 + 1/a)^l = 1 - b_1 + b_2 - b_3 + \cdots$. Again assume the theorem is false. The error ϵ_n is less than b_{n+1} and is greater than $b_{n+1} - b_{n+2}$. So

$$\epsilon_n < b_{n+1} = \left|\binom{l}{n+1}\frac{1}{a^{n+1}}\right|$$

$$\left|\binom{l}{n+2}\frac{1}{a^{n+2}} - \binom{l}{n+3}\frac{1}{a^{n+3}}\right| = b_{n+2} - b_{n+3} < \epsilon_{n+1}.$$

Therefore if the theroem is false

$$\left|\binom{l}{n+2}\frac{1}{a^{n+2}} - \binom{l}{n+3}\frac{1}{a^{n+3}}\right| < C\binom{l}{n+1}^2\left(\frac{1}{a^{n+1}}\right)^2$$

$$\left|\frac{l-n-1}{n+2}\cdot\frac{a-\dfrac{l-n-2}{n+3}}{a^2}\right| < C\left|\binom{l}{n+1}\frac{1}{a^{n+1}}\right|.$$

This last inequality cannot hold for all n, for the left member approaches a constant greater than zero, $(a \neq 1)$, and the right member approaches zero since it is a term of the series. Therefore we have a contradiction. Q.E.D.

Corollary: *There are sequences defined by the difference equation* (48) *which converge more rapidly to* $\beta^{1/p}$ *than the sequence of partial sums of any binomial series.*

University of Maryland
College Park, Maryland

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SOME GENERAL APPROXIMATION THEOREMS

BY LAWRENCE M. GRAVES

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1. Introduction. In this paper comparatively simple proofs are given for some general theorems on the approximation of a given function by "smoother" functions. Well known theorems on approximation by polynomials, by trigonometric sums, and by integral means are special cases. The proofs, given in Section 3, are so formulated that there is little more complication in the case of functions of m variables than in the case of functions of one variable. Theorems on the term-by-term integration and differentiation of the approximating sequence are included. In Section 4 certain cases of iteration of the approximating transformations are considered. In Section 5 it is shown that when the given function is continuous and vanishes on a closed point set E, certain types of approximating functions may be required to vanish on a neighborhood of E while possessing as many continuous derivatives as may be derived.

Let A denote the interval $0 \leqq x_i \leqq a_i$, $(i = 1, \cdots, m)$, of m-dimensional space, and let B denote the interval $-a_i \leqq x_i \leqq a_i$. Let $\{Q_n(x)\}$ be an infinite sequence of bounded measurable functions defined on B and let (k_n) be an infinite sequence of positive numbers. Then if $f(x)$ is a function defined and integrable on A, we may set

$$\begin{aligned} P_n(x;f) &= k_n \int_A f(z)Q_n(z-x)\,dz \\ &= k_n \int_0^{a_1} \cdots \int_0^{a_m} f(z)Q_n(z-x)\,dz_m \cdots dz_1. \end{aligned} \tag{1:1}$$

It is observed at once that if the function $Q_n(x)$ is of class $C^{(r)}$ on B then P_n is of class $C^{(r)}$ on A, and if $Q_n(x)$ is a polynomial so is P_n. For the most part we shall wish to assume that the numbers k_n are related to the functions Q_n by the formula

$$1/k_n = \int_B Q_n(x)\,dx = \int_{-a_1}^{a_1} \cdots \int_{-a_m}^{a_m} Q_n\,dx_m \cdots dx_1. \tag{A}$$

Interesting examples of sequences $Q_n(x)$ are:

(i) $Q_n(x) = [1 - \sum_i x_i^2]^n$, with $\sum_i a_i^2 \leqq 1$;

(ii) $Q_n(x) = \prod_i (1 - x_i^2)^n$, with $a_i \leqq 1$;

(iii) $Q_n(x) = \prod_i \cos^n x_i$, with $a_i \leqq \pi/2$;

(iv) $Q_n(x) = 1$ for $|x_i| \leqq 1/n$, $Q_n(x) = 0$ elsewhere;

(v) $Q_n(x) = \prod_i (1/n^2 - x_i^2)^n$ for $|x_i| \leqq 1/n$, $Q_n(x) = 0$ elsewhere.

In examples (i) and (ii) the polynomials $P_n(x)$ related to a function f by formula (1:1) are frequently called Stieltjes polynomials. In example (iii) we obtain trigonometric polynomials $P_n(x)$. In example (iv) the functions $P_n(x)$ are integral means of $f(x)$ over hyper-cubes.

We shall wish to consider the following properties which sequences (Q_n) and (k_n) may possess. We let S_ϵ denote the interior of the sphere with center at the origin and radius ϵ.

(B) *Each $Q_n(x)$ is bounded and measurable and non-negative on the domain B, and positive on a subset of B of positive measure.*

(C) *For every $\epsilon > 0$, $\lim_n k_n Q_n(x) = 0$ uniformly on $B - S_\epsilon$.*

(D) *Each Q_n is of class $C^{(r)}$ on B.*

(E) *If R_n denotes a partial derivative of Q_n, of total order less than r, then for every $\epsilon > 0$, $\lim_n k_n R_n(x) = 0$ uniformly on $B - S_\epsilon$.*

When the numbers k_n are given by formula (A), examples (i), (ii) and (iii) possess all these properties, and so does example (*v*) for $n > n_r$ sufficiently great. Example (iv) has properties (*B*) and (*C*). To verify property (*E*) for example (i), we notice that if R_n is a derivative of Q_n of order μ, then R_n is $O[n^\mu(1 - \epsilon^2)^n]$ uniformly on $B - S_\epsilon$. Also, with $\delta = n^{-1/2}$,

$$1/k_n \geqq \int_{S_\delta} [1 - \sum x_i^2]^n \, dx \geqq \int_{S_\delta} [1 - 1/n]^n \, dx = bn^{-m/2}[1 - 1/n]^n,$$

where b is a properly chosen constant, so that k_n is $O(n^{m/2})$.

In case the function f is originally defined only on a subset A_0 of A, its domain of definition may be extended to be the whole of A in a variety of ways. If A_0 is the interval $b_i \leqq x_i \leqq c_i$, we may set

$$\begin{aligned} &f(x_1, x_2, \cdots, x_m) = f(b_1, x_2, \cdots, x_m), \\ &(0 \leqq x_1 < b_1, \quad b_i \leqq x_i \leqq c_i, \quad i = 2, \cdots, m), \end{aligned} \tag{1:2}$$

and so on until the extension is completed. With this extension the properties of continuity, of m-tuple absolute continuity, and of satisfying a Lipschitz condition are all preserved. In place of (1:2) we might use

$$f(x_1, x_2, \cdots, x_m) = f(2b_1 - x_1, x_2, \cdots, x_m). \tag{1:3}$$

This extension preserves also the property of absolute continuity in the sense on Tonelli. If f is integrable on a measurable subset A_0 of A, we may set $f = 0$ on the complement of A_0. For an extension preserving uniform continuity or a Lipschitz condition when A_0 is an arbitrary subset of A, the reader is referred to McShane[1] [10] or Whitney [11, p. 63, footnote]. When f is of

[1] Numbers in brackets refer to the bibliography at the end.

class $C^{(r)}$ (in a suitable sense) on a closed set A_0, an extension preserving this property has been given by Whitney [11]. A simpler method of extension applicable in special cases has been devised by Hestenes, but is not yet published.

We shall find it convenient throughout the proofs to set $f = 0$ outside the interval A. Then by making the change of variables $z_i = x_i + v_i$ we find that for x in A the formula (1:1) may be written

$$P_n(x; f) = k_n \int_B f(x + v)Q_n(v)\, dv. \tag{1:4}$$

2. Preliminary lemmas. The class of all measurable functions f such that $|f|^p$ is integrable on A is denoted as usual by L^p, for $1 \leqq p < \infty$. For this class set

$$\|f\|_p = \left[\int_A |f|^p\, dx\right]^{1/p}. \tag{2:1}$$

We make the convention that L^∞ denotes the class of essentially bounded and measurable functions, and for this class denote by $\|f\|_\infty$ the essential upper bound of $|f(x)|$ on A. Then if $p + q = pq$ ($q = \infty$ when $p = 1$) we have the Hölder inequality

$$\left|\int_A f(x)g(x)\, dx\right| \leqq \|f\|_p \cdot \|g\|_q \tag{2:2}$$

holding when f is in L^p and g is in L^q, and the triangle inequality[2]

$$\|f + g\|_p \leqq \|f\|_p + \|g\|_p \tag{2:3}$$

holding when f and g are both in L^p.

We shall have need for two lemmas, which are readily proved.

LEMMA 1. *Let $f(x)$ be in L^p, ($1 \leqq p < \infty$), and set $f_h(x) = f(x + h)$. Then $\|f_h - f\|_p$ approaches zero with h.*

To prove this note that by the triangle inequality (2:3), $\|f_h - f\|_p \leqq \|f_{1h} - f_1\|_p + \|f_{2h} - f_2\|_p$ when $f = f_1 + f_2$, and $\|f_h - f\|_p \leqq 2\|f\|_p$. From these inequalities it follows that the class E of functions f for which the conclusion of the lemma holds is linear and closed in L^p. The class E obviously contains the characteristic functions of sub-intervals of A. The class of all linear combinations of such characteristic functions is dense[3] on L^p, and hence $E = L^p$.

LEMMA 2. *If $f(x)$ is in L^p, ($1 \leqq p \leqq \infty$), on A, then $f(x + v)$ as a function of $2m$ variables, x in A and v in B, is also in L^p.*[4]

To prove this we note first that if a set E is closed in A, then the set E^* of points (x, v) of the product space AB for which $x + v$ is in E is likewise closed

[2] See, e.g., Hobson [8], vol. I, p. 588.

[3] See, e.g., Hobson [8], vol. II, p. 251.

[4] Compare Cinquini [6], pp. 59–62.

in AB. Consequently if E is open in A, E^* is open in AB. If E is an interval with measure μE, then E^* has measure

$$\mu E^* = \mu E \cdot \prod a_i . \tag{2:4}$$

Since every open set may be represented as a sum of non-overlapping closed intervals, it follows that (2:4) holds for sets E open in A, and hence for sets E closed in A. By applying the criterion that a set E is measurable if and only if for every $\epsilon > 0$ there exists a closed set E_1 and an open set E_2 such that $E_1 \subset E \subset E_2$ and $\mu(E_2 - E_1) < \epsilon$, we see that when E is measurable E^* is also measurable, and (2:4) still holds. Hence $f(x + v)$ is measurable on AB. Let ${}_N(x) = |f(x)|$ where $|f(x)| \leqq N$, $f_N(x) = N$ where $|f(x)| > N$. Then

$$\int_A \int_B [f_N(x + v)]^p \, dv \, dx = \int_A \int_A [f_N(v)]^p \, dv \, dx$$

$$= \int_A [f_N(v)]^p \, dv \cdot \prod_i a_i \leqq \int_A |f(v)|^p \, dv \cdot \prod a_i .$$

This with Fubini's theorem[5] implies the desired conclusion.

3. The general approximation theorems. For the sake of completeness, we include the following theorem, whose proof is an immediate extension of that given by Landau [1].

THEOREM 1. *Suppose the sequences (Q_n) and (k_n) have properties (A), (B), and (C). Let A_0 be a closed set interior to A, and suppose that $f(x)$ is bounded and measurable on A, and continuous at the points of A_0. Then $P_n(x)$ converges uniformly to $f(x)$* on A_0.

Let $\alpha > 0$ be the minimum distance from A_0 to the boundary of A. For an arbitrary $\delta > 0$ there exists a positive $\epsilon < \alpha$ such that

$$|f(x + v) - f(x)| < \delta \tag{3:1}$$

whenever x is in A_0 and $\sum_i v_i^2 < \epsilon^2$. Now from formula (1:4) and property (A) we have

$$|P_n(x) - f(x)| = k_n \left| \int_B [f(x + v) - f(x)] \, Q_n(v) \, dv \right|$$

$$\leqq k_n \int_{S_\epsilon} |f(x + v) - f(x)| \, Q_n(v) \, dv \tag{3:2}$$

$$+ k_n \int_{B - S_\epsilon} [|f(x + v)| + |f(x)|] \, Q_n(v) \, dv.$$

The first term on the right is less than δ for x in A_0 by (3:1) and properties (A) and (B), and the second term approaches zero with $1/n$ uniformly in x, by property (C).

[5] Saks [9], p. 77, Hobson [8], vol. 1, p. 577.

THEOREM 2. *Suppose the sequences* (Q_n) *and* (k_n) *have properties* (A), (B), *and* (C). *Let* $f(x)$ *be in* L^p, $(1 \leqq p < \infty)$. *Then*

(a) $\lim_n \int_A | P_n(x) - f(x) |^p \, dx = 0$;

(b) *a subsequence* $P_{n_j}(x)$ *converges to* $f(x)$ *almost everywhere on* A;

(c) *the integrals* $\int_E | P_n(x) |^p \, dx$ *are equi-absolutely continuous;*

(d) *if* M *is the essential upper bound of* $| f(x) |$, *then* $| P_n(x) | \leqq M$ *on* A;

(e) *if* $g(x)$ *is in* L^q $(p + q = pq, 1 \leqq p \leqq \infty)$,

$$\lim_n \int_A | g(x) | \, | P_n(x) - f(x) | \, dx = 0.$$

By the use of the first equality in (3:2) and the Hölder inequality we have

$$\begin{aligned} | P_n(x) - f(x) |^p &= k_n^p \left| \int_B [f(x + v) - f(x)] Q_n(v) \, dv \right|^p \\ &\leqq k_n^p \int_B | f(x + v) - f(x) |^p Q_n(v) \, dv \left[\int_B Q_n(v) \, dv \right]^{p-1} \\ &= k_n \int_B | f(x + v) - f(x) |^p Q_n(v) \, dv. \end{aligned}$$

From this by Lemma 2 and Fubini's theorem on interchange of order of integration we obtain

$$\begin{aligned} \int_A | P_n(x) - f(x) |^p \, dx &\leqq k_n \int_B \int_A | f(x + v) - f(x) |^p Q_n(v) \, dx \, dv \\ &= k_n \int_{S_\epsilon} \int_A | f(x + v) - f(x) |^p \, dx \, Q_n(v) \, dv \\ &\quad + k_n \int_{B - S_\epsilon} \int_A | f(x + v) - f(x) |^p \, dx \, Q_n(v) \, dv. \end{aligned}$$

The first term on the right converges to zero with ϵ, uniformly with respect to n, by Lemma 1 and properties (A) and (B). The second term converges to zero with $1/n$ for each ϵ, by property (C). This proves part (a) of the conclusion. Part (b) is a consequence of (a).[6] Part (c) follows by simple arguments from the triangle inequality

$$\left[\int_E | P_n(x) |^p \, dx \right]^{1/p} \leqq \left[\int_E | P_n(x) - f(x) |^p \, dx \right]^{1/p} + \left[\int_E | f(x) |^p \, dx \right]^{1/p},$$

part (a), and the absolute continuity of $\int_E | P_n(x) |^p \, dx$ and $\int_E | f(x) |^p \, dx$.

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[6] See Hobson [8], vol. II, pp. 239–45.

Part (d) is an immediate consequence of (1:4). In case p is finite, part (e) follows at once from the Hölder inequality and part (a). In case f is bounded and g merely integrable, we may make an indirect proof, supposing that

$$\limsup \int_A | g(x) | | P_n(x) - f(x) | dx = \beta > 0.$$

Then for a properly selected subsequence (P_{n_j}) we have

$$\lim \int_A | g(x) | | P_{n_j}(x) - f(x) | dx = \beta,$$

$$\lim P_{n_j}(x) = f(x) \text{ almost everywhere,}$$

$$| g(x) | | P_{n_j}(x) - f(x) | \leqq 2M | g(x) | \text{ almost everywhere.}$$

This leads to a contradiction with the theorem on term-by-term integration for Lebesgue integrals.

LEMMA 3. *Suppose that the sequences* (Q_n) *and* (k_n) *have properties* (C) *and* (D) *with* $r \geqq 1$. *Let* $f(x)$ *be absolutely continuous in* x_1 *for almost all* $(x_2, \cdots, x_m)$, *and let* f *and the partial derivative* f_{x_1} *be integrable on* A. *Suppose also that* $f(0, x_2, \cdots, x_m)$ *is integrable. Let* A_0 *be a closed set interior to* A. *Then* $\lim_n | P_n(x; f_{x_1}) - P_{nx_1}(x; f) | = 0$ *uniformly in* A_0.

Since the functions Q_n are of class C′, differentiation under the integral sign is permissible in formula (1:1), and we have

$$P_{nx_1}(x; f) = -k_n \int_A f(z) Q_{nz_1}(z - x) \, dz.$$

Hence by Fubini's theorem and the absolute continuity of f in x_1 we have

$$\begin{aligned} P_n(x; f_{x_1}) - P_{nx_1}(x; f) &= k_n \int_A [f_{x_1}(z) Q_n(z - x) + f(z) Q_{nz_1}(z - x)] \, dz \\ &= k_n \int_0^{a_2} \cdots \int_0^{a_m} [f(z) Q_n(z - x)]_{z_1=0}^{z_1=a_1} \, dz_m \cdots dz_2. \end{aligned} \tag{3:3}$$

Also

$$\int_A f_{x_1}(x) \, dx = \int_0^{a_2} \cdots \int_0^{a_m} [f(x)]_{x_1=0}^{x_1=a_1} \, dx_m \cdots dx_2,$$

so that the integrability of $f(0, x_2, \cdots, x_m)$ implies that of $f(a_1, x_2, \cdots, x_m)$. Thus (3:3) with property (C) justifies the desired conclusion.

Let us recall that a function $f(x_1 \cdots x_m)$ of several variables is absolutely continuous on A in the sense of Tonelli in case it is continuous in $(x_1, \cdots, x_m)$, absolutely continuous in each x_i for almost all values of the remaining variables, and each partial derivative f_{x_i} is integrable on A.

THEOREM 3. *Suppose the sequences* (Q_n) *and* (k_n) *have properties* (A), (B), (C), *and* (D) *for* $r \geqq 1$. *Let* $f(x)$ *be absolutely continuous in the sense of Tonelli, and*

let the partial derivative f_{x_i} be in L^p, $(1 \leqq p < \infty)$. Let A_0 be a closed set interior to A. Then

(a) $\lim_n \int_{A_0} | P_{nx_i}(x) - f_{x_i}(x) |^p \, dx = 0$;

(b) *a subsequence of (P_{nx_i}) converges to f_{x_i} almost everywhere on A;*

(c) *for $E \subset A_0$, the integrals $\int_E | P_{nx_i}(x) |^p \, dx$ are equi-absolutely continuous;*

(d) *if f satisfies a Lipschitz condition on A, the partial derivatives P_{nx_i} are uniformly bounded on A_0;*

(e) *if f_{x_i} is continuous at the points of A_0, the sequence (P_{nx_i}) converges to f_{x_i} uniformly on A_0;*

(f) *if $g(x)$ is in L^q $(p + q = pq, 1 \leqq p \leqq \infty)$,*

$$\lim_n \int_{A_0} | g(x) | \, | P_{nx_i}(x) - f_{x_i}(x) | \, dx = 0.$$

To prove part (a) we use the triangle inequality

$$\left[\int_{A_0} | P_{nx_i}(x;f) - f_{x_i}(x) |^p \, dx\right]^{1/p} \leqq \left[\int_{A_0} | P_n(x;f_{x_i}) - f_{x_i}(x) |^p \, dx\right]^{1/p} + \left[\int_{A_0} | P_{nx_i}(x;f) - P_n(x;f_{x_i}) |^p\right]^{1/p}.$$

The first term on the right approaches zero by Theorem 2 with f replaced by f_{x_i}, and the second term approaches zero by Lemma 3. To prove (b) we may consider a sequence of closed intervals A_0 interior to A but with boundaries approaching the boundary of A. The existence of the desired subsequence converging almost everywhere on A is then secured by the well-known diagonal method. Parts (c) and (f) follow from (a) as in the proof of Theorem 2. Part (d) follows from Theorem 2 and Lemma 3, and part (e) from Theorem 1 and Lemma 3.

THEOREM 4. *Let the sequences (Q_n) and (k_n) have properties* (A), (B), (C), *and* (D) *and* (E) *for $r \geqq m$. Suppose that $f(x)$ is absolutely continuous*[7] *in the set of variables $(x_1, \cdots, x_m)$, and in addition that when one or more of the variables x_i is set equal to zero $f(x)$ is absolutely continuous in the remaining set of variables. Let $t(x)$ denote a partial derivative of $f(x)$ in which the derivative with respect to any one variable x_i occurs to at most the first order, and let $T_n(x)$ denote the corresponding partial derivative of $P_n(x)$. Let $t(x)$ be in L^p, $(1 \leqq p < \infty)$, and let A_0 be a closed set interior to A. Then*

(a) $\lim_n \int_{A_0} | T_n(x) - t(x) |^p \, dx = 0$;

(b) *a subsequence of (T_n) converges to t almost everywhere on A;*

(c) *for $E \subset A_0$ the integrals $\int_E | T_n(x) |^p \, dx$ are equi-absolutely continuous;*

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[7] That is, m-tuply absolutely continuous.

(d) *if t is bounded on A, the sequence (T_n) is uniformly bounded on A_0*;

(e) *if t is continuous at the points of A_0, the sequence (T_n) converges to t uniformly on A_0.*

(f) *the functions $P_n(x)$ are equi-absolutely continuous in the set of variables $(x_1, \cdots, x_m)$ on A_0.*

We shall carry through the proof for the case of two independent variables. The method of extending the proof to other cases and of obtaining various modifications of Theorem 4 will then be apparent. The hypothesis on the function $f(x_1, x_2)$ implies that it may be represented in the form

$$f(x_1, x_2) = f(0, 0) + \int_0^{x_1} f_{x_1}(x_1, 0)\, dx_1 + \int_0^{x_2} f_{x_2}(0, x_2)\, dx_2 + \int_0^{x_2}\int_0^{x_1} f_{x_2 x_1}\, dx_2\, dx_1 .$$

It is easily seen that the hypotheses of Theorem 3 are verified, so that the desired properties of P_{nx_1} and P_{nx_2} follow from that theorem. Now the hypotheses of Lemma 3 are verified with f replaced by f_{x_2}, so that

$$\lim_n | P_n(x; f_{x_2 x_1}) - P_{nx_1}(x; f_{x_2}) | = 0 \tag{3:4}$$

uniformly on A_0. Also the hypotheses of Lemma 3 are verified with Q_n replaced by $R_n = Q_{nx_1}$. Now

$$P_{nx_1}(x; f_{x_2}) = k_n \int_A f_{x_2}(z) R_n(z - x)\, dz$$

so that by Lemma 3,

$$\lim_n | P_{nx_1}(x; f_{x_2}) - P_{nx_1 x_2}(x; f) | = 0 \tag{3:5}$$

uniformly on A_0. Combining (3:4) and (3:5) we have

$$\lim_n | P_n(x; f_{x_2 x_1}) - P_{nx_1 x_2}(x; f) | = 0$$

uniformly on A_0. From this and Theorems 2 and 1 the conclusions of the theorem for $t(x) = f_{x_2 x_1}$ then follow as in the proof of Theorem 3. The final conclusion (f) follows at once from (c) with $p = 1$ and $T_n(x) = P_{nx_1 x_2}$.

4. Iteration of the approximating transformations. In the preceding discussion we have considered the linear transformations $P_n(f)$ given by formula (1:1), which transform certain normed linear spaces $\mathcal{F} = [f]$ into subsets of themselves. In certain cases these transformations are also equi-continuous, that is, there exists a constant M such that

$$|| P_n(f) || \leqq M \, || f || \tag{4:1}$$

for every n and every f in $\mathfrak{F}$. They may also have the property that

$$\lim_{n} \| P_n(f) - f \| = 0 \tag{4:2}$$

for every f in $\mathfrak{F}$. When the sequence (P_n) has the properties (4:1) and (4:2) and (K_n) is another sequence of transformations, not necessarily linear, but having at least property (4:2) it follows that

$$\lim_{q,\, n} \| P_q K_n f - f \| = 0. \tag{4:3}$$

Thus when the transformations K_n are also linear and have property (4:1), the composite transformations $K_n P_n$ have the same properties (4:1) and (4:2) as the factors have. In particular the iterates P_n^s, $(s = 2, 3, \cdots)$, have properties (4:1) and (4:2).

We now proceed to discuss some cases to which the argument of the preceding paragraph is applicable.

Case I. The space $\mathfrak{F}$ consists of all functions f continuous on a closed rectangular region A_0 interior to A, and $\| f \|$ is the maximum of $| f(x) |$ on A_0. We extend the definition of f over the remainder of A as indicated by (1:2). Then by properties (A) and (B) and the formula (1:4), $\| P_n(f) \| \leqq \| f \|$, so that (4:1) holds true with $M = 1$. The condition (4:2) holds by Theorem 1. Note that before another transformation K_n of the same kind is applied the functions $P_n(x; f)$ are to be regarded at first as defined only for x in A_0, and then the same extension of definition over the remainder of A is to be used as was used for f itself. This procedure seems to be necessary in order that the preceding arguments may be validly applied. In examples (iv) and (v) it is easily seen that this precaution is unnecessary for many applications of the preceding result.

The iteration of the transformations P_n seems to be of special interest when the $P_n(f)$ are the integral means of f, that is, they are defined by the Q_n of example (iv). For in this case when the function f is of class $C^{(r)}$ on the region A_0, the transformed function $P_n(f)$ is of class $C^{(r+1)}$ on an arbitrary closed subregion A_1 interior to A_0, for n sufficiently large, and hence $P_n^s(f)$ is of class $C^{(r+s)}$ on an arbitrary closed subregion A_s interior to A_0, for n sufficiently large.

Case II. The space $\mathfrak{F}$ consists of all functions f defined and of class L^p, $(1 \leqq p < \infty)$, on a fixed measurable subset A_0 of A, and

$$\| f \| = \left[\int_{A_0} | f |_p \, dx \right]^{1/p}.$$

In this case we agree to set $f = 0$ on the complement of A_0. Then by the same argument as that used at the beginning of the proof of Theorem 2, we find

$$\int_A | P_n(x) |^p \, dx \leqq k_n \int_B \int_A | f(x + v) |^p \, dx \, Q_n(v) \, dv \leqq k_n \int_B \| f \|^p Q_n(v) \, dv = \| f \|^p,$$

so that (4:1) holds, again with $M = 1$. Theorem 2 yields the property (4:2).

If we take A_0 to be the whole of A and the transformations $P_n(f)$ to be the integral means of f, the iterated transformations P_n^s yield approximating functions of class $C^{(s-1)}$ at least on an arbitrary closed subregion A_1 interior to A, for n sufficiently large.

Case III. The space $\mathcal{F}$ consists of all functions f absolutely continuous in the sense of Tonelli on a closed interval A_0 interior to A, and

$$\|f\| \equiv \underset{\substack{x \text{ in } A_0 \\ i=1, \cdots, m}}{\text{maximum}} \left\{ |f(x)|, \int_{A_0} |f_{x_i}|\, dx \right\}.$$

We extend the definition of f over A as indicated by (1:3), and let M denote a constant sufficiently large to satisfy all the conditions imposed on it. As in case I we find $|P_n(x;f)| \leqq \|f\|$, and as in the proof of Lemma 3 we obtain

$$\begin{aligned} &\int_{A_0} |P_{nx_1}(x;f) - P_n(x;f_{x_1})|\, dx \\ &= k_n \int_{A_0} \left| \int_0^{a_2} \cdots \int_0^{a_m} [f(z)Q_n(z-x)]_{z_1=0}^{z_1=a_1} dz_m \cdots dz_2 \right| dx \\ &\leqq k_n \|f\| \int_A \int_0^{a_2} \cdots \int_0^{a_m} [Q_n(z-x)|^{z_1=a_1} + Q_n(z-x)|^{z_1=0}]\, dz_m \cdots dz_2\, dx \\ &\leqq 2\|f\| \cdot a_2 \cdots a_m \leqq M\|f\|/2. \end{aligned} \tag{4:4}$$

By case II we have

$$\int_{A_0} |P_n(x;f_{x_i})|\, dx \leqq \int_A |f_{x_i}|\, dx \leqq M\|f\|/2,$$

and combining this inequality with (4:4) (with x_1 replaced by x_i), yields

$$\int_{A_0} |P_{nx_i}(x;f)|\, dx \leqq M\|f\|,$$

so that (4:1) holds. The condition (4:2) follows from Theorems 1 and 3. We note that in this case example (iv) must be ruled out, since it does not satisfy the hypotheses of Theorem 3.

In case the function f satisfies a Lipschitz condition, and $L(f)$ denotes the minimum Lipschitz constant for f on the interval A_0, it follows from the proofs of Lemma 3 and Theorem 3 that there is a constant M such that

$$L(P_n(f)) \leqq M[L(f) + \max |f(x)|]. \tag{4:5}$$

If (K_n) is another sequence of transformations having property (4:5) and such that $|K_n(x;f)| \leqq M \cdot \max |f(x)|$, then

$$L(P_q K_n f) \leqq M^2[L(f) + 2\max |f(x)|],$$

so that the double sequence of functions $P_q K_n f$ satisfies for each f a uniform Lipschitz condition.

5. Approximating functions vanishing where f vanishes.[8] In this section we suppose that the function f is continuous on the closed interval A_0 interior to A, and vanishes on the closed set A_1 in A_0. We may agree to extend the domain of definition of f to be the whole of A by the method indicated in (1:2) or by that indicated in (1:3). Consider the sequence of transformations K_n defined as follows:

$$\begin{aligned} K_n(x; f) &= f(x) - 1/n \text{ where } f(x) \geqq 1/n, \\ &= f(x) + 1/n \text{ where } f(x) \leqq -1/n, \\ &= 0 \qquad\qquad \text{ where } -1/n \leqq f(x) \leqq 1/n. \end{aligned}$$

Let the transformations P_n correspond to the Q_n of example (v). Then for each n there is an integer q_n sufficiently great so that the function $P_{q_n}K_nf$ vanishes on a neighborhood of the set A_1, and is of class $C^{(r)}$ on A.

In case I of Section 4 we find that the K_n satisfy (4:2) so that

$$\lim_{q,n} \| P_qK_nf - f \| = 0, \tag{5:1}$$

$$\lim_{n} \| P_{q_n}K_nf - f \| = 0. \tag{5:2}$$

In case III, where the function f is supposed to be absolutely continuous in the sense of Tonelli, we consider first the case when $f \geqq 0$. Then each partial derivative f_{x_i} vanishes wherever it exists on the set where f vanishes. Thus $K_{nx_i} = f_{x_i}$ almost everywhere on the set where $f = 0$ and almost everywhere on the set where $f > 1/n$, and $| K_{nx_i} | \leqq | f_{x_i} |$, $\lim_n K_{nx_i} = f_{x_i}$ almost everywhere. Since each function absolutely continuous in the sense of Tonelli is representable in a standard way as the difference of two non-negative functions having the same property, we find

$$\lim_n \int_{A_0} | K_{nx_i}(x; f) - f_{x_i}(x) | \, dx = 0,$$

so that (4:2) holds in this case also. Consequently we obtain (5:1) and (5:2) for this case, by the results of Section 4.

When the function f satisfies a Lipschitz condition, the transformed functions K_nf obviously satisfy the same Lipschitz condition, and hence by the last paragraph of Section 4 the functions P_qK_nf satisfy a uniform Lipschitz condition.

The results of this section show that the following theorem is valid.

THEOREM 5. *Let $f(x)$ be continuous on a bounded closed set A_0, and vanish on a set $A_1 \subset A_0$. Then there exists a sequence of functions $\varphi_n(x)$, each of class $C^{(r)}$ on the whole space and vanishing on a neighborhood of the set A_1, such that $\lim_n \varphi_n(x) = f(x)$ uniformly on A_0. In case the set A_0 is an interval and the*

[8] Cf. Reid [12], p. 859.

function f is absolutely continuous in the sense of Tonelli, the sequence (φ_n) may be required to satisfy also the condition that

$$\lim_n \int_{A_0} | \phi_{nx_i} - f_{x_i} | \, dx = 0.$$

When the function f satisfies a Lipschitz condition on the set A_0, the functions φ_n may be required to satisfy also a uniform Lipschitz condition on A_0.

Note that the last statement is valid when the set A_0 is an arbitrary bounded set, since a Lipschitz function may have its domain of definition extended over the whole of space without losing that property.

THE UNIVERSITY OF CHICAGO.

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TAUBERIAN CONDITIONS[1]

BY RALPH PALMER AGNEW

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1. Introduction. It is the object of this paper to introduce several new Tauberian classes and to discuss their relations to each other and to various Tauberian classes which have been previously considered.

As the title indicates, our interest lies in Tauberian conditions themselves rather than in Tauberian theorems involving them. In order that our discussion of Tauberian theorems may be as free as possible from complications which would draw attention from the Tauberian conditions, we consider in this paper only Tauberian theorems for the C_1 (Cesàro of order 1, or arithmetic mean) method of summability. That the fundamental Fourier transform method of Wiener[2] can be applied to give general Tauberian theorems for the classes T, T^*, T'_ρ, and T' which we shall define is shown by a paper of H. R. Pitt to which we refer in §10. It is probable that the methods of Wiener and Pitt apply also to the more general classes T''_B of §11. Each of the classic C_1 Tauberian theorems to which we refer in §2 is a corollary of Theorem 9.1.

2. Classic Tauberian theorems. Let $u_1 + u_2 + \cdots$ be a series (convergent or divergent) of complex terms, and let

$$s_n = \sum_{k=1}^{n} u_k; \qquad \sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k \qquad\qquad n = 1,2,\cdots$$

be respectively the sequence of partial sums and the C_1 transform of $\sum u_n$. An elementary *Abelian theorem* states that if $s_n \to s$, then $\sigma_n \to s$; in other words, that C_1 is a regular method of summability. That $\sigma_n \to \sigma$ does not imply $s_n \to \sigma$ is illustrated by the series $\sum (-1)^{n+1}$ for which $\sigma_n \to \frac{1}{2}$. If, however, $\sigma_n \to \sigma$ and the terms of $\sum u_n$ satisfy certain auxiliary *Tauberian conditions*, then $s_n \to \sigma$. The following, in which $\tilde{T}$ denotes a class of series, is a typical Tauberian theorem.

2.1 *If $\sigma_n \to \sigma$ and $\sum u_n \in \tilde{T}$, then $s_n \to \sigma$.*

The essential part of the conclusion is that s_n converges; for if it is known that s_n converges, then regularity of C_1 implies that $s_n \to \sigma$.

If $\tilde{T}$ is the class T_C of convergent series, then 2.1 is true but trivial since the class T is too small to make 2.1 significant. If $\tilde{T}$ is the union of the class T_C and the class T_D of all divergent (non-convergent) series not summable C_1, then again 2.1 is obviously true; but again the theorem is trivial since the defini-

[1] Presented to the American Mathematical Society, February 24, 1940.

[2] N. Wiener, Tauberian Theorems, Annals of Math., vol. 33 (1932), pp. 1–100.

tion of $\tilde{T}$ requires that one know whether s_n converges *before* one can decide whether $\sum u_n$ satisfies the hypothesis of 2.1. If 2.1 is to be useful, there must naturally be some series for which it is easier to determine that $\sum u_n \epsilon \tilde{T}$ than it is to determine whether $\sum u_n$ converges. For application to a given series $\sum u_n$ belonging to some Tauberian class $\tilde{T}$, the best theorems are those involving $\tilde{T}$ for which it is easiest to show that $\sum u_n \epsilon \tilde{T}$. These considerations indicate that there probably is no single "best" Tauberian class which will eventually supplant all others.

A simple Tauberian theorem states that if $\sigma_n \to \sigma$ and $\sum u_n$ satisfies the condition $nu_n \to 0$ of Tauber[3] then $s_n \to \sigma$. A more general Tauberian theorem asserts that if $\sigma_n \to \sigma$ and $\sum u_n$ satisfies the condition $n \mid u_n \mid < K$ of Hardy[4] then $s_n \to \sigma$. A unilateral Tauberian theorem states that if $\sigma_n \to \sigma$ and $\sum u_n$ is a real series satisfying the condition $nu_n < K$ (or the condition $nu_n > -K$) of Landau[5] then $s_n \to \sigma$. The last theorem was extended by Lukacs[6] to cover complex series, by the next theorem in which S' denotes the set of points in the complex plane which lie in a sector with vertical angle less than π. If $\sigma_n \to \sigma$ and $nu_n \epsilon S'$, then $s_n \to \sigma$. The conditions $nu_n \to 0$, $n \mid u_n \mid < K$, $nu_n < K$, and $nu_n \epsilon S'$ (as well as some conditions of Schmidt which we discuss in §7) are known as *order conditions.* The Tauberian condition of the next theorem is known as a *gap condition.*[7] If $\sigma_n \to \sigma$ and $u_n = 0$ when $n \neq n_1$, $n_2, \cdots$, where $n_1 < n_2 < \cdots$ is a sequence of indices for which $\liminf_{p\to\infty} n_{p+1}/n_p > 1$, then $s_n \to \sigma$. "Gap Tauberian theorems" such as the above have been called[8] remarkable since there is no order condition on the terms u_n.

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[6] F. Lukacs, Bemerkung zu einem Konvergenzsatze des Herrn Landau, Arch. d. Math. u. Phys. (3) 23 (1915), pp. 367–378. For further results and references, see L. S. Bosqnquet and M. L. Cartwright, Mathematische Zeitschrift, vol. 37 (1933), pp. 416–423; and N. Wiener, ibid., vol. 36 (1933), pp. 787–789.

[7] See Hardy and Littlewood, A further note on the converse of Abel's theorem, Proceedings of the London Mathematical Society, Series 2, vol. 25 (1926), pp. 219–236. This fundamental paper removes order conditions from the hypotheses of Tauberian theorems due to Landau (1907) and Littlewood (1910). For a recent and brief proof and discussion of gap theorems, see A. E. Ingham, On the high-indices theorem of Hardy and Littlewood, Quarterly Journal of Math., Oxford Series, vol. 8 (1937), pp. 1–7.

[8] For a recent example, see N. Levinson, General Gap Tauberian Theorems I, Proceedings of the London Mathematical Society, Series 2, vol. 44 (1938), pp. 289–306, p. 289.

3. The class T. Let K be a positive constant, and let θ be an angle for which $0 < \theta < \pi/2$. Corresponding to each real angle ψ, let $S(\psi) \equiv S(K, \theta, \psi)$ denote the "sector with vertical angle $< \pi$" consisting of all points z of the complex plane representable in the form

$$z = -Ke^{i\psi} + \rho e^{i(\psi+\varphi)}$$

where $\rho \geqq 0$ and $-\theta \leqq \varphi \leqq \theta$. The sector $S(\psi)$ has its vertex at the point $-Ke^{i\psi}$; the angle at the vertex of the sector is 2θ; the half-line bisecting the sector passes through the origin and makes an angle ψ with the positive real axis; and each sector $S(\psi)$ may be obtained by rotating the particular sector $S(0)$ about the origin. The sectors $S(\psi)$ are special sectors in that the bisectors pass through the origin; however if S' is any sector with vertical angle $< \pi$, then it is easy to determine K, θ, and ψ so that S' is a subset of $S(K, \theta, \psi)$.

DEFINITION 3.1. *A series $\sum u_n$ will be said to belong to class T if K, θ, λ, and $\psi_1, \psi_2, \psi_3, \cdots$ exist such that $K > 0$, $0 < \theta < \pi/2$, $\lambda > 1$, and, for each sufficiently great index k,*

$$nu_n \in S(\psi_k) \equiv S(K, \theta, \psi_k) \qquad k \leqq n < \lambda k. \tag{3.11}$$

The Tauberian condition $nu_n \in S'$ of §2 implies that there is a *fixed* sector S' which contains all elements of the sequence nu_n; hence the condition $nu_n \in S'$ may be termed a *fixed-sector condition.* On the other hand the Tauberian condition $\sum u_n \in T$ allows *diffierent* sectors S to contain different aggregates of consecutive elements of the sequence nu_n; hence the condition $\sum u_n \in T$ may be termed a *variable-sector condition.*

A real series $\sum u_n$ satisfies the fixed-sector condition $nu_n \in S'$ if and only if it satisfies at least one of the two unilateral conditions $nu_n < K$ and $nu_n > -K$. The condition $\sum u_n \in T$ does not imply either of these unilateral conditions. In fact it is easy to show that a real series $\sum u_n$ is in class T if and only if $K > 0$ and $\lambda > 1$ exist such that for each sufficiently great k, say $k > k_0$, either

$$nu_n < K \qquad k \leqq n < \lambda k \tag{3.12}$$

or

$$nu_n > -K \qquad k \leqq n < \lambda k. \tag{3.13}$$

The essential point here is that (3.12) may hold for a certain set of values of $k > k_0$ while (3.13) holds for the remaining values of $k > k_0$.

4. Some special series in class T. The next two theorems show that if $\sum u_n$ satisfies one of the Tauberian conditions of §2, then $\sum u_n \in T$.

THEOREM 4.1. *If $nu_n \in S'$, where S' is some sector of the complex plane with vertical angle $< \pi$, then $\sum u_n \in T$.*

This is obvious since we can choose a sector $S(K, \theta, \psi)$ such that $S' \subset S(K, \theta, \psi)$ and then take $\psi_k = \psi$ for each $k = 1, 2, \cdots$ to show that $\sum u_n \in T$. It

follows that if $\sum u_n$ satisfies one of the other order conditions of §2, then $\sum u_n \in T$.

THEOREM 4.2. *If $n_1 < n_2 < \cdots$ is a gap sequence such that*

$$\liminf_{p\to\infty} n_{p+1}/n_p > 1,$$

and $u_n = 0$ when $n \neq n_1, n_2, \cdots$, then $\sum u_n \in T$.

Choose $\lambda > 1$ such that $\liminf n_{p+1}/n_p > \lambda$, and then choose an index P such that

$$n_{p+1} > \lambda n_p \qquad p \geqq P.$$

For each $k \geqq n_p$ there is at most one index n in the interval $k \leqq n < \lambda k$ for which $u_n \neq 0$. We can assume that $u_n \neq 0$ for an infinite set of n for otherwise certainly $\sum u_n \in T$. For each $k = 1, 2, \cdots$ let ψ_k be the amplitude φ_n of $u_n = \rho_n \exp i\varphi_n$ where $\rho_n > 0$, $-\pi < \varphi_n \leqq \pi$, and n is the least index such that $n \geqq k$ and $u_n \neq 0$. Then, however $K > 0$ and $0 < \theta < \pi/2$ are determined, the relation

$$nu_n \in S(K, \theta, \psi_k) \qquad k \leqq m < \lambda k$$

holds when $k \geqq k_P$ and accordingly $\sum u_n \in T$. The series satisfying the hypothesis of Theorem 4.2 are gap series with no condition whatever on u_n when n has one of the values $n_1, n_2, n_3, \cdots$.

It is possible to generalize Theorem 4.2 in various ways; for example, the following theorem is an immediate consequence of Theorems 4.2 and 6.1.

THEOREM 4.3. *If $n_1 < n_2 < \cdots$ is a gap sequence as in Theorem 4.2 and*

$$n \mid u_n \mid < K \qquad n \neq n_1, n_2, \cdots$$

then $\sum u_n \in T$.

Each series $\sum u_n$ satisfying the hypothesis of Theorem 4.3 is representable in the form $\sum (u_n + v_n)$ where each one of the series $\sum u_n$ and $\sum v_n$ satisfies a classic Tauberian condition of §2. That T contains important classes of series not so representable is illustrated by the fact that T contains each ordinary Dirichlet series with non-negative coefficients.

THEOREM 4.4. *If $a_n \geqq 0$ then $\sum a_n n^{-z} \in T$ for each complex z.*

Setting $u_n = a_n n^{-z}$ and $z = x + iy$ where x and y are real, we find that $nu_n = A_n \exp i\varphi_n$ where $A_n = na_n n^{-x} \geqq 0$ and $\varphi_n = -y \log n$. If $0 < \theta < \pi/2$, and $\lambda > 1$ is chosen such that $\mid y \mid \log \lambda < \theta$, then

$$\mid \varphi_n - \varphi_k \mid \leqq \mid y \mid \log \lambda < \theta \qquad k \leqq n < \lambda k$$

and it follows that $\sum u_n \in T$. In case the argument z in the Dirichlet series $\sum a_n n^{-z}$ is pure imaginary say $z = iy$, the hypothesis $a_n \geqq 0$ can be relaxed (see Theorem 6.2) to give

THEOREM 4.5. *If $\sum a_n \in T$, then $\sum a_n n^{-iy} \in T$ for each real y.*

In particular, if a_n is real and $a_n < K$ (or $a_n > -K$), then $\sum a_n n^{-z} \in T$ for each z on the line $\Re z = 1$.

It would be possible to mention other interesting subclasses of T; we have merely given enough to indicate that Tauberian theorems which are proved for the class T have diverse applications.

5. Characterizations of T. The following two quite trivial theorems give characterizations of T which are useful in proofs of Tauberian theorems involving T.

THEOREM 5.1. *In order that $\sum u_n \in T$, it is necessary and sufficient that $K > 0$, $0 < \theta < \pi/2$, $\alpha < 1 < \beta$, and $\psi_1, \psi_2, \cdots$ exist such that for each sufficiently great index k*

$$nu_n \in S(\psi_k) \equiv S(K, \theta, \psi_k) \qquad \alpha k < n < \beta k. \tag{5.11}$$

Sufficiency is obvious. To prove necessity, let $\sum u_n \in T$ and choose $K > 0$, $0 < \theta < \pi/2$, $\lambda > 1$, and $\psi_1', \psi_2', \cdots$ such that for each sufficiently great k

$$nu_n \in S(K, \theta, \psi_k') \qquad k \leqq n < \lambda k. \tag{5.12}$$

If in (5.12) we replace k by $k' = [k\lambda^{-1/2}]$ and choose α and β such that

$$\lambda^{-1/2} < \alpha < 1 < \beta < \lambda^{1/2} \tag{5.13}$$

we find that, for k sufficiently great,

$$nu_n \in S(K, \theta, \psi_{k'}') \qquad \alpha k < n < \beta k. \tag{5.14}$$

Thus (5.11) holds when ψ_k is $\psi_{k'}'$ and the theorem is proved.

THEOREM 5.2. *In order that $\sum u_n \in T$, it is necessary and sufficient that $K > 0$, $0 < \theta < \pi/2$, $\mu < 1$, and $\psi_1, \psi_2, \cdots$ exist such that for each sufficiently great index k*

$$nu_n \in S(\psi_k) \equiv S(K, \theta, \psi_k) \qquad \mu k < n \leqq k. \tag{5.21}$$

Necessity is implied by Theorem 5.1. Proof that (5.21) implies (5.11), and hence $\sum u_n \in T$, differs very little from our proof that (5.12) implies (5.11).

6. Properties of T. The definition 3.1 of T obviously implies that if $\sum u_n \in T$ and $u_n' = u_n$ for all sufficiently great n, then $\sum u_n' \in T$. In particular if $u_1 + u_2 + u_3 + \cdots \in T$, then $(u_1 - A) + u_2 + u_3 + \cdots \in T$ for each complex number A.

It is easy to show that if $\sum \Re u_n \in T$ and $\sum \Im u_n \in T$, then $\sum u_n \in T$; that is, that $\sum u_n$ belongs to class T if the series of real parts and the series of pure imaginary parts belong to the class. However simple examples show that $\sum u_n \in T$ implies neither $\sum \Re u_n \in T$ nor $\sum \Im u_n \in T$. The class T is not linear. For example, if $u_n' = (-1)^n + 1$ and $u_n'' = (-1)^n - 1$, then $\sum u_n' \in T$ and $\sum u_n'' \in T$; but $\sum (u_n' + u_n'') \in T$ fails.

THEOREM 6.1. *If $\sum u_n \in T$ and $n \mid v_n \mid < K_1$, then $\sum (u_n + v_n) \in T$.*

Choose $K > 0$, $0 < \theta < \pi/2$, $\lambda > 1$, and $\psi_1, \psi_2, \cdots$, and an index k_0 such that when $k > k_0$

$$nu_n \in S(K, \theta, \psi_k) \qquad k \leqq n < \lambda k. \tag{6.11}$$

Elementary computation shows that

$$n(u_n + v_n) \in S(K + K_1 \csc \theta, \theta, \psi_k) \qquad k \leqq n < \lambda k, \tag{6.12}$$

and hence that $\sum (u_n + v_n) \in T$.

THEOREM 6.2. *If $\sum u_n \in T$ and $\beta_1, \beta_2, \cdots$ is a real sequence such that*

$$\lim_{\lambda \to 1+} \limsup_{k \to \infty} \max_{k \leqq n < \lambda k} |\beta_n - \beta_k| = 0 \tag{6.21}$$

then $\sum u_n \exp i\beta_n \in T$.

Obtaining (6.11) as above, we have, when $k > k_0$ and $k \leqq n < \lambda k$

$$nu_n = -Ke^{i\psi_k} + \rho_n e^{i(\psi_k + \varphi_n)} \tag{6.22}$$

where $\rho_n \geqq 0$ and $|\varphi_n| \leqq \theta < \pi/2$. Choose $\theta' > 0$ such that $\theta'' \equiv \theta + \theta' < \pi/2$. Let λ be decreased and let k_0 be increased if necessary so that, when $k > k_0$ and $k \leqq n < \lambda k$, (6.22) holds and also $|\beta_n - \beta_k| < \theta'$. If we set $\psi_k' = \psi_k + \beta_k$ and $\varphi_n' = \varphi_n + \beta_n - \beta_k$ (our notation does not take into account the fact that φ_n' as well as ρ_n and φ_n depend on k) we obtain

$$nu_n e^{i\beta_n} = -Ke^{i(\psi_k + \beta_n)} + \rho_n e^{i(\psi_k' + \varphi_n')} \tag{6.23}$$

where $\rho_n \geqq 0$ and $|\varphi_n'| \leqq |\varphi_n| + |\beta_n - \beta_k| \leqq \theta''$. It can be shown that $nu_n \exp i\beta_n \in S(K \csc \theta'', \theta'', \psi_k')$ when $k > k_0$ and $k \leqq n < \lambda k$ and Theorem 6.2 is proved.

THEOREM 6.3. *If $\sum u_n \in T$ and $a_1, a_2, \cdots$ is a bounded sequence of real non-negative constants, then $\sum a_n u_n \in T$.*

Obtaining (6.22) as above, we find that

$$na_n u_n = -Ka_n e^{i\psi_n} + a_n \rho_n e^{i(\psi_k - \varphi_n)}.$$

If K' is the least upper bound of the non-negative numbers Ka_n, then

$$na_n u_n = -K'e^{i\psi_n} + \rho_n' e^{i(\psi_k - \varphi_n')}$$

where $\rho_n' \geqq 0$ and $-\theta \leqq \varphi_n' \leqq \theta$; hence $\sum a_n u_n \in T$.

Theorem 6.3 can be used to show that if $\sum u_n \in T$, then each series obtained by adjoining or removing a finite set of terms at the beginning of the series is also in class T; if q is a positive integer and one of the two series

$$u_1 + u_2 + u_3 + \cdots$$

$$u_q + u_{q+1} + u_{q+2} + \cdots$$

is in class T, so also is the other

7. The class T^*. The developments of this section are largely extensions of the ideas of "langsam oszillierend" and "langsam abfallende" sequences introduced by Schmidt.[9]

It is well known that if $\sum u_n$ satisfies the Tauberian condition $n \mid u_n \mid < K$ then the simple estimate,

$$|s_q - s_p| \leqq \sum_{n=p+1}^{q} |u_n| \leqq K \sum_{n=p+1}^{q} n^{-1}$$
$$\leqq K(\log q - \log p + o_k) \leqq K \log (\beta/\alpha) + o_k,$$

in which $\alpha k < p < q < \beta k$ and $o_k \to 0$ as $k \to \infty$, shows that the partial sums s_n of $\sum u_n$ satisfy the Tauberian condition

$$\lim_{\alpha\to 1-,\, \beta\to 1+} \limsup_{k\to\infty} \operatorname*{maximum}_{\alpha k<p<q<\beta k} |s_q - s_p| = 0. \tag{7.01}$$

Since

$$\operatorname*{maximum}_{k\leqq n<\beta k} |s_n - s_k| \leqq \operatorname*{maximum}_{\alpha k<p<q<\beta k} |s_q - s_p|$$

when $\alpha < 1 < \beta$, the condition (7.01) implies that

$$\lim_{\lambda\to 1+} \limsup_{k\to\infty} \operatorname*{maximum}_{k\leqq n<\lambda k} |s_n - s_k| = 0; \tag{7.011}$$

and it is easy to show that (7.011) implies (7.01).

It is likewise well known that if $\sum u_n$ is a real series satisfying the unilateral condition $nu_n > -K$, then a similar computation shows that

$$\lim_{\alpha\to 1-,\, \beta\to 1+} \liminf_{k\to\infty} \operatorname*{minimum}_{\alpha k<p<q<\beta k} (s_q - s_p) \geqq 0. \tag{7.02}$$

If (7.02) holds, then obviously

$$\lim_{\lambda\to 1+} \liminf_{k\to\infty} \operatorname*{minimum}_{k\leqq n<\lambda k} (s_n - s_k) \geqq 0. \tag{7.021}$$

If (7.021) holds, then corresponding to each $\epsilon > 0$ there exists $\lambda_0 > 1$ such that when $1 < \lambda < \lambda_0$ there exists $k_0 = k_0(\lambda)$ such that $(s_n - s_k) > -\epsilon$ when $k > k_0$ and $k \leqq n < \lambda k$. If $1 < \lambda < \lambda_0$ and $\lambda^{-1/2} < \alpha < 1 < \beta < \lambda^{1/2}$ and $\alpha k < p \leqq r \leqq q < \beta k$ then, for k sufficiently great, $(s_r - s_p) > -\epsilon$ and $(s_q - s_r) > -\epsilon$ so that $(s_q - s_p) > -2\epsilon$; and hence we can obtain (7.02). Thus the Tauberian conditions (7.02) and (7.021) are equivalent. Schmidt (loc. cit., p. 136) has called a real series "langsam abfallend" when its partial sums s_n have the following property: If $q = q(p)$ is a sequence of indices such that $q \geqq p$ for each $p = 1, 2, \cdots$ and $q/p \to 1$ as $p \to \infty$, then $\liminf_{p\to\infty}(s_q - s_p) \geqq 0$. Using this definition, it is easy to show that the class of "langsam abfallend" series is identical with the class of series satisfying (7.021), and hence also identical with the class of series satisfying (7.02).

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[9] Robert Schmidt, Über divergente Folgen und lineare Mittelbildungen, Mathematische Zeitschrift, vol. 22 (1925), pp. 89–152; p. 132 and p. 136.

In a similar manner, it can be shown that if a real series $\sum u_n$ satisfies the Tauberian condition $nu_n < K$, then $\sum u_n$ satisfies the equivalent Tauberian conditions

$$\lim_{\alpha\to 1-,\beta\to 1+} \limsup_{k\to\infty} \operatorname*{maximum}_{\alpha k<p<q<\beta k} (s_q - s_p) \leqq 0. \tag{7.03}$$

and

$$\lim_{\lambda\to 1+} \limsup_{k\to\infty} \operatorname*{maximum}_{k\leqq n<\lambda k} (s_n - s_k) \leqq 0. \tag{7.031}$$

That (7.02) and (7.03) are essentially different conditions is evidenced by the fact that the sequence $s_n = n$ satisfies (7.02) but not (7.03), and the sequence $s_n = -n$ satisfies (7.03) but not (7.02).

Numerous Tauberian theorems involving series satisfying such conditions as (7.01) $\cdots$ (7.031) can be found in a paper of Szasz,[10] a monograph of Karamata,[11] and references given by these authors. A double series analogue of (7.02) has been used by Knopp[12] to obtain Tauberian theorems for double series.

We proceed to show that if $\sum u_n \epsilon T$, then the partial sums of $\sum u_n$ satisfy a condition more general than (7.01), (7.02), and (7.03). Let $\sum u_n \epsilon T$. Then $K > 0$, $0 < \theta < \pi/2$, $\alpha_0 < 1 < \beta_0$, and $-\psi_1, -\psi_2, \cdots$ exist such that for each sufficiently great k

$$nu_n = -Ke^{-i\psi_k} + \rho_n e^{i(-\psi_k+\varphi_n)} \qquad \alpha_0 k < n < \beta_0 k \tag{7.04}$$

where $\rho_n \geqq 0$ and $|\varphi_n| \leqq \theta$. If $\alpha_0 < \alpha < 1 < \beta < \beta_0$, then for each real δ_k

$$nu_n e^{i(\psi_k+\delta_k)} = -Ke^{i\delta_k} + \rho_n e^{i(\varphi_n+\delta_k)} \tag{7.05}$$

when $\alpha k < n < \beta k$. Let

$$\Delta = \pi/2 - \theta. \tag{7.06}$$

If $|\delta_k| \leqq \Delta$, then (7.05) and the inequalities $\rho_n \geqq 0$ and $|\varphi_n + \delta_k| \leqq \theta + \Delta = \pi/2$ imply that

$$\mathcal{R} nu_n e^{i(\psi_k+\delta_k)} \geqq -K \qquad \alpha k < n < \beta k. \tag{7.07}$$

Hence, when $|\delta_k| \leqq \Delta$ and $\alpha k < p < q < \beta k$,

$$\begin{aligned}\mathcal{R}(s_q - s_p)e^{i(\psi_k+\delta_k)} &= \sum_{n=p+1}^{q} \mathcal{R} u_n e^{i(\psi_k+\delta_k)} \geqq -K \sum_{n=p+1}^{q} \frac{1}{n} \\ &\geqq -K \log(\beta/\alpha) + o_k\end{aligned} \tag{7.08}$$

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[10] O. Szasz, Converse theorems of summability for Dirichlet's series, Trans. American Math. Soc., 39 (1936), pp. 117–130.

[11] J. Karamata, Sur les theoremes inverses des procedes de sommabilite, Hermann and Cie, Paris (1937), 47 pp.

[12] K. Knopp, Limitierungs—Umkehrsätze für Doppelfolgen, Math. Zeit., vol. 45 (1939), pp. 573–589.

where o_k denotes a quantity which, when α and β are fixed, converges to 0 as $k \to \infty$.

Setting

$$(7.09) \qquad F_1(\alpha, \beta, k, \delta_k, \psi_k) = \underset{\alpha k < p < q < \beta k}{\text{mimimum}} \, \mathcal{R}\,(s_q - s_p)\, e^{i(\psi_k + \delta_k)}$$

we see that when $|\delta_k| \leqq \Delta$

$$(7.10) \qquad F_1(\alpha, \beta, k, \delta_k, \psi_k) \geqq -K \log(\beta/\alpha) + o_k.$$

Setting

$$(7.11) \qquad F_2(\alpha, \beta, k) = \max_{-\infty < \psi_k < \infty} \; \min_{|\delta_k| \leqq \Delta} F_1(\alpha, \beta, k, \delta_k, \psi_k)$$

we see that

$$(7.12) \qquad F_2(\alpha, \beta, k) \geqq -K \log(\beta/\alpha) + o_k;$$

hence

$$(7.13) \qquad \liminf_{k \to \infty} F_2(\alpha, \beta, k) \geqq -K \log(\beta/\alpha)$$

and finally, since the left member of (7.13) is monotone increasing as $\alpha \to 1-$, $\beta \to 1+$,

$$(7.14) \qquad \lim_{\alpha \to 1-,\, \beta \to 1+} \; \liminf_{k \to \infty} F_2(\alpha, \beta, k) \geqq 0.$$

Definition 7.2. *Let T^* denote the class of series $\sum u_n$ whose partial sums satisfy the condition: a constant Δ exists such that*[13] $0 < \Delta < \pi/2$ *and*

$$(7.21) \qquad \begin{gathered} \lim_{\alpha \to 1-,\, \beta \to 1+} \; \liminf_{k \to \infty} \; \underset{-\infty < \psi_k < \infty}{\text{maximum}} \\ \underset{|\delta_k| \leqq \Delta}{\text{minimum}} \; \underset{\alpha k < p < q < \beta k}{\text{minimum}} \; \mathcal{R}(s_q - s_p) e^{i(\psi_k + \delta_k)} \geqq 0. \end{gathered}$$

Since (7.21) is merely the result of substituting for $F_2(\alpha, \beta, k)$ in (7.14), and we have shown that (7.14) is implied by the hypothesis that $\sum u_n \in T$, we have proved

Theorem 7.3. *If $\sum u_n \in T$, then $\sum u_n \in T^*$.*

The condition (7.21) would be unchanged in meaning if the subscript k were removed from ψ_k and δ_k; we retain the subscript because it serves to emphasize the meaning of (7.21). It is the "variable sector" feature of the class T which accounts for the presence of the ψ's in (7.21), and it is the presence of the ψ's in (7.21) which gives the class T its generality. It should of course be observed that, in the parade of limits and extrema in (7.12), the operation of taking the maximum over $-\infty < \psi_k < \infty$ is the only one which strives to make the inequality sign in (7.12) run the right way. The condition that $\sum u_n \in T^*$ (as well as the condition $\sum u_n \in T$) may be stated roughly as follows: in the

[13] The left member of (7.21) is a monotone increasing function of Δ; hence if (7.21) holds for some $\Delta > 0$, we can suppose $0 < \Delta < \pi/2$.

infinite succession of "jumps" in the complex plane from s_1 to s_2 to s_3 to $\cdots$, jumps which are advanced in the sequence and too close together must not be both too large and in directions too nearly opposite.

8. Properties of T^*. It is easy to show that if $\sum u_n$ satisfies one of the conditions (7.01), (7.02), and (7.03), then $\sum u_n \epsilon T^*$. That T^* includes series $\sum u_n$ for which (7.01), (7.02), and (7.03) all fail is illustrated by large classes of series and in particular by many real gap series. That T^* contains series not in T is illustrated by the fact that T^* contains each convergent series while T does not. For example, if $s_n = (-1)^n/(n+1)^{1/2}$, then $\sum u_n$ is not in the class T. It is apparent from (7.21) that if $u_1 + u_2 + \cdots$ is a series in T^*, then $(u_1 - A) + u_2 + \cdots$ is a series in T^* for each complex constant A.

It is possible to characterize T^* by conditions similar to but different from (7.21). If we replace ψ_k by $\psi_k + \pi$ in (7.21) and remove the factor $e^{i\pi} = -1$, we see that $\sum u_n \epsilon T^*$ if and only if $\Delta > 0$ exists such that

$$\begin{aligned}(8.01)\qquad &\lim_{\alpha\to 1-,\,\beta\to 1+}\ \limsup_{k\to\infty}\ \underset{-\infty<\psi_k<\infty}{\text{minimum}} \\ &\underset{|\delta_k|\leqq\Delta}{\text{maximum}}\ \underset{\alpha k<p<q<\beta k}{\text{maximum}}\ \mathcal{R}(s_q - s_p)e^{i(\psi_k+\delta_k)} \leqq 0.\end{aligned}$$

It thus appears that, while (7.02) and (7.03) are essentially different conditions, (7.21) and (8.01) differ only in appearance.

THEOREM 8.1. *In order that $\sum u_n \epsilon T^*$, it is necessary and sufficient that $\Delta > 0$ exist such that*

$$\begin{aligned}(8.11)\qquad &\lim_{\lambda\to 1+}\ \liminf_{k\to\infty}\ \underset{-\infty<\psi_k<\infty}{\text{maximum}} \\ &\underset{|\delta_k|<\Delta}{\text{minimum}}\ \underset{k\leqq n<\lambda k}{\text{minimum}}\ \mathcal{R}(s_n - s_k)e^{i(\psi_k+\delta_k)} \geqq 0.\end{aligned}$$

Necessity is a consequence of the fact that if $\alpha < 1 < \beta = \lambda$, and $R_{n,k}$ is a real double sequence, then

$$(8.12)\qquad \underset{k\leqq n<\lambda k}{\text{minimum}}\ R_{n,k} \geqq \underset{\alpha k<p<q<\beta k}{\text{minimum}}\ R_{p,q}.$$

To prove sufficiency, suppose (7.21) fails. Then, when $\Delta > 0$, a constant $H = H(\Delta) > 0$ exists such that the left member of (7.21) is less than $-H$. Hence $\alpha_0 < 1 < \beta_0$ exist such that, when α and β are fixed numbers satisfying the inequality $\alpha_0 < \alpha < 1 < \beta < \beta_0$, the inequality

$$(8.13)\qquad \underset{-\infty<\psi_k<\infty}{\text{maximum}}\ \underset{|\delta_k|\leqq\Delta}{\text{minimum}}\ \underset{\alpha k<p<q<\beta k}{\text{minimum}}\ \mathcal{R}(s_q - s_p)e^{i(\psi_k+\delta_k)} < -H$$

holds for an infinite set of values of the index k, say $k_1 < k_2 < \cdots$. When k has a fixed one of the values $k_1, k_2, \cdots$ there exist indices p_k and q_k such that

$$(8.14)\qquad \alpha k < p_k < q_k < \beta k$$

and

$$(8.15)\qquad \underset{-\infty<\psi<\infty}{\text{maximum}}\ \underset{|\delta|<\Delta}{\text{minimum}}\ \mathcal{R}(s_{q_k} - s_{p_k})e^{i(\psi+\delta)} < -H.$$

Since (8.14) implies that $p_k \leqq q_k < \lambda p_k$ where $\lambda = \beta/\alpha > 1$, we can show that (8.15) contradicts (8.11) and Theorem 8.1 is proved.

The condition (8.11) is more convenient than (7.21) when one wishes to show that $\sum u_n \in T^*$; but the condition (7.21) which we have featured in the definition of T^* is more convenient than (8.11) when the condition $\sum u_n \in T^*$ is used as a part of the hypothesis of a Tauberian theorem.

Our definition and discussion of T^* naturally applies to series $\sum u_n$ whose terms are real. Since much of the existing Tauberian theory applies only to series with real terms (though in many cases there is an immediate application to series of complex terms of which the real and imaginary parts separately satisfy the Tauberian conditions involved) it may be of interest to see that the condition (7.21) can be thrown into a form somewhat simpler when $\sum u_n$ is real. Using the definition 7.2 of T^*, we can prove

THEOREM 8.2. *If the terms of $\sum u_n$ are real, then $\sum u_n \in T^*$ if and only if*

$$\lim_{\alpha\to 1-,\, \beta\to 1+} \limsup_{k\to\infty} \max_{j_k=1,2} \operatorname*{minimum}_{\alpha k<p<q<\beta k} (-1)^{j_k}(s_q - s_p) \geqq 0. \tag{8.21}$$

It can also be shown that (8.21) may be replaced in Theorem 8.2 by the condition

$$\lim_{\lambda\to 1+} \limsup_{k\to\infty} \max_{j_k=1,2} \operatorname*{minimum}_{k\leqq n<\lambda k} (-1)^{j_k}(s_n - s_k) \geqq 0. \tag{8.22}$$

When the conditions (8.21) and (8.22) are made more restrictive by removal of the expressions involving j, they reduce to the characterizations (7.02) and (7.021) of the "langsam abfallend" series of Schmidt.

9. Tauberian theorems for C_1 summability. We now prove the two following theorems of which the first is a Tauberian convergence theorem and the second is a Tauberian oscillation theorem.

THEOREM 9.1. *If $\sigma_n \to \sigma$ and $\sum u_n \in T^*$, then $s_n \to \sigma$.*

THEOREM 9.2. *If $0 < \Delta < \pi/2$, $\epsilon > 0$, $0 < \alpha < 1 < \beta < \infty$, $k_0 > 0$, and $\psi_1, \psi_2, \cdots$ are such that when $k > k_0$,*

$$\operatorname*{minimum}_{\alpha k<p<q<\beta k} \mathcal{R}(s_q - s_p)e^{i(\psi_k+\delta_k)} \geqq -\epsilon \qquad |\delta_k| \leqq \Delta, \tag{9.21}$$

then for each complex number A,

$$\limsup_{n\to\infty} |s_n - A| \leqq [\epsilon + C \limsup_{n\to\infty} |\sigma_n - A|] \operatorname{cosec} \Delta \tag{9.22}$$

where C is the greater of the numbers $(1 + \alpha)/(1 - \alpha)$ and $(\beta + 1)/(\beta - 1)$.

The first theorem is obviously a corollary of the second, for if $\sum u_n \in T^*$ and $\sigma_n \to \sigma$, we can set $A = \sigma$ and choose ϵ as near 0 as we please. We give a direct proof of Theorem 9.2 which is similar to (and not essentially more difficult than) standard proofs of Tauberian theorems for classes smaller than T^*. Let $k' = [\alpha k]$ and $k'' = [\beta k]$. Then for each sufficiently great k the identity

$$s_1 + s_2 + \cdots + s_k = (s_1 + s_2 + \cdots + s_{k'}) + (s_{k'+1} + \cdots + s_k) \tag{9.23}$$

can be written in the form

$$k\sigma_k = k'\sigma_{k'} + (k - k')s_k - \sum_{p=k'+1}^{k} (s_k - s_p),$$

and division by $k - k'$ gives

$$s_k = U'_k + V'_k \tag{9.25}$$

where

$$U'_k = \frac{k\sigma_k - k'\sigma'_k}{k - k'}; \qquad V'_k = \frac{1}{k - k'} \sum_{p=k'+1}^{k} (s_k - s_p). \tag{9.26}$$

If we set $L = \limsup |\sigma_n|$, then

$$\limsup |U'_k| \leqq \limsup [(k|\sigma_k| + k'|\sigma_{k'}|)/(k - k')]$$
$$\leqq L \lim [(k + k')(k - k')] = L(1 + \alpha)/(1 - \alpha) \leqq CL.$$

Using (9.21) gives for $k > k_0$ and $|\delta_k| \leqq \Delta$

$$\mathcal{R} V'_k e^{i(\psi_k + \delta_k)} \geqq \frac{1}{k - k'} \sum_{p=k'+1}^{k} (-\epsilon) = -\epsilon.$$

Hence

$$\liminf_{k\to\infty} \mathcal{R} s_k e^{i(\psi_k+\delta_k)} \geqq -\epsilon - CL. \tag{9.27}$$

Starting with the identity obtained by replacing k by k'' and k' by k in (9.23), a similar argument shows that

$$\limsup_{k\to\infty} \mathcal{R} s_k e^{i(\psi_k+\delta_k)} \leqq \epsilon + CL.$$

Hence

$$\limsup_{k\to\infty} |\mathcal{R} s_k e^{i(\psi_k+\delta_k)}| \leqq \epsilon + CL.$$

If we set $\varphi_k = \arg s_k$ so that $s_k = |s_k| \exp i\varphi_k$, this becomes

$$\limsup_{k\to\infty} |s_k| |\cos(\varphi_k + \psi_k + \delta_k)| \leqq \epsilon + CL.$$

As δ_k ranges over the interval $-\Delta \leqq \delta_k \leqq \Delta$, the angle $(\varphi_k + \psi_k + \delta_k)$ must assume a value which differs from each odd multiple of $\pi/2$ by at least Δ; hence

$$\limsup_{k\to\infty} |s_k| \cos(\pi/2 - \Delta) \leqq \epsilon + CL$$

and

$$\limsup |s_n| \leqq [\epsilon + C \limsup |\sigma_n|] \operatorname{cosec} \Delta. \tag{9.28}$$

If s_n is replaced by $s_n - A$, then (9.21) still holds and σ_n is replaced by $\sigma_n - A$; therefore (9.28) holds when s_n and σ_n are replaced by $s_n - A$ and $\sigma_n - A$ respectively, and Theorem 9.2 is proved.

10. The Tauberian class of Pitt. In §9 we gave what is perhaps the simplest "direct" proof of C_1 Tauberian theorems for the class T^*. A more indirect proof of C_1 Tauberian theorems for the class T^* leads naturally to the general Tauberian classes of Pitt[14] and to generalizations of them.

Let $\sum u_n \in T^*$ and let $\epsilon > 0$. Then there exist $0 < \alpha < 1 < \beta$, a sequence $\psi_1, \psi_2, \cdots$, and an index k_0 such that when $k > k_0$ and $|\delta_k| \leqq \Delta$

$$\mathcal{R}(s_q - s_p)e^{i(\psi_k+\delta_k)} > -\epsilon \qquad \alpha k < p < q < \beta k.$$

In particular, when $k > k_0$ and $|\delta_k| \leqq \Delta$,

$$\mathcal{R}(s_p - s_k)e^{i(\psi_k+\delta_k)} > -\epsilon \qquad k < p < \beta k \tag{10.01}$$

and

$$\mathcal{R}(s_k - s_p)e^{i(\psi_k+\delta_k)} > -\epsilon \qquad \alpha k < p < k. \tag{10.02}$$

Our next step is to let A denote an arbitrary complex number; to replace s_p by $s_p - A$ and s_k by $s_k - A$ in (10.01) and (10.02); and then to set $\varphi_k = \arg(s_k - A)$ (our notation does not take into account the fact that φ_k depends on A) so that (10.01) and (10.02) become respectively

$$(s_p - A)e^{i(\psi_k+\delta_k)} > -\epsilon + |s_k - A| \cos(\varphi_k + \psi_k + \delta_k) \qquad k < p < \beta k$$

and

$$(s_p - A)e^{i(\psi_k+\delta_k+\pi)} > -\epsilon - |s_k - A| \cos(\varphi_k + \psi_k + \delta_k) \qquad \alpha k < p < k.$$

As δ_k ranges over the interval $-\Delta \leqq \delta_k \leqq \Delta$, the angle $(\varphi_k + \psi_k + \delta_k)$ must assume a value ω_k differing in absolute value from each odd multiple of $\pi/2$ by at least Δ; if possible, choose δ_k such that $|\delta_k| \leqq \Delta$ and ω_k lies in the first or fourth quadrants, and then set $\theta_k = \psi_k + \delta_k$; otherwise choose δ_k such that $|\delta_k| \leqq \Delta$ and ω_k lies in the second or third quadrants, and then set $\theta_k = \psi_k + \delta_k + \pi$. These choices of δ_k and θ_k give

$$\mathcal{R}(s_p - A)e^{i\theta_k} > -\epsilon + |s_k - A| \cos(\pi/2 - \Delta) \tag{10.03}$$

for each p in at least one of the two ranges $k < p < \beta k$ and $\alpha k < p < k$.

Let, for each ρ in the interval $0 < \rho \leqq 1$, T'_ρ denote the class of series $\sum u_n$ whose partial sums s_n satisfy the following condition: corresponding to each $\epsilon > 0$, there exist numbers α and β such that $\alpha < 1 < \beta$, a sequence $\theta_1, \theta_2, \cdots$, and an index k_0 such that when $k > k_0$, the inequality

$$\mathcal{R}s_p e^{i\theta_k} > -\epsilon + \rho |s_k| \tag{10.04}$$

holds for each p in at least one of the two ranges $\alpha k < p < k$ and $k < p < \beta k$. The argument leading to (10.04) establishes the following theorem in which Δ is the constant involved in the definition of T^*.

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[14] H. R. Pitt, General Tauberian Theorems, Proc. London Math. Soc. (2) vol. 44 (1938), pp. 243-288. We refer to this paper as G.T.T. To compare the classes of G.T.T. with ours, it is necessary to associate a sequence s_n with a step function $s(x)$ defined by $s(x) = s_1 + s_2 + \cdots + s_{[x]}$ and make an exponential change of variable.

THEOREM 10.1. *If a series $\sum u_n$ with partial sums s_n is in class T^*, then for each constant A the series $(u_1 - A) + u_2 + \cdots$ with partial sums $s_n - A$ is in the class T'_ρ for which $\rho = \cos(\pi/2 - \Delta)$.*

It is clear that if $\rho' < \rho$, then $T_{\rho'} \supset T_\rho$. Let T' denote the union of the classes T'_ρ for $0 < \rho \leqq 1$. Then $T^* \subset T'$. If we modify the definition involving (10.04) by allowing ρ to depend on ϵ, we obtain a class P of series which is identical[15] with the class of series satisfying condition T of G.T.T. pp. 244–245.

DEFINITION 10.2. *Let P denote the class of series $\sum u_n$ whose partial sums s_n satisfy the following condition: corresponding to each $\epsilon > 0$ there exist numbers ρ, α, and β such that $\rho > 0$ and $\alpha < 1 < \beta$; a real sequence $\theta_1, \theta_2, \cdots$; and an index k_0 such that when $k > k_0$ the inequality*

$$\mathcal{R} s_p e^{i\theta_k} > -\epsilon + \rho \mid s_k \mid \tag{10.21}$$

holds for each p in at least one of the two ranges $\alpha k < p < k$ and $k < p < \beta k$.

Obviously $T' \subset P$. Hence the relations $T \subset T^* \subset T'$ and $T'_\rho \subset T'$ imply that each Tauberian oscillation theorem of G.T.T. which applies to series in P applies a fortiori to series in the classes T, T^*, T'_ρ, and T'.

11. The classes T''_E and T''. We now define still more general Tauberian classes by removing the requirement that (10.21) hold for each p in a range having k at one of the extremes. However we maintain (by restricting the sequences α_n and β_n) the requirement that (10.21) shall hold for each p in a range which is neither too short nor too remote from k. It will be clear that $P \subset T'' \subset T''_E$ for each $E > 0$.

DEFINITION 11.1. *Corresponding to each $E > 0$, let T''_E denote the class of series $\sum u_n$ whose partial sums s_n satisfy the following condition: there exist a constant $\rho > 0$; sequences α_k and β_k such that $0 < \alpha_k < \beta_k$, $k\alpha_k \to \infty$, and*

$$\limsup_{k\to\infty} (\beta_k + \alpha_k)/(\beta_k - \alpha_k) \equiv F < \infty; \tag{11.11}$$

a real sequence θ_k; and an index k_0 such that when $k > k_0$

$$\mathcal{R} s_p e^{i\theta_k} > -E + \rho \mid s_k \mid \qquad k\alpha_k \leqq p \leqq k\beta_k. \tag{11.12}$$

Let T'' denote the intersection of the classes T''_E for $E > 0$.

THEOREM 11.2. *If $\sum u_n \in T''_E$ then*

$$\limsup_{k\to\infty} \mid s_k \mid \leqq [E + F \limsup_{n\to\infty} \mid \sigma_n \mid]\rho^{-1} \tag{11.21}$$

where s_k and σ_n are respectively the partial sums and the C_1 transform of $\sum u_n$, and the constants F and $\rho > 0$ are any set for which (11.11) *and* (11.12) *hold.*

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[15] It is easy to see that, apart from the difference in scale of the independent variable, the definitions differ only in appearance.

Using the notation of the definition of T''_E, let $k' = [k\alpha_k]$ and $k'' = [k\beta_k]$. Then the identity

$$(11.22) \qquad (s_1 + \cdots + s_{k''}) - (s_1 + \cdots + s_{k'}) = s_{k'+1} + \cdots + s_{k''}$$

leads, for k sufficiently great, to

$$(11.23) \qquad \begin{aligned} \mathcal{R}(k''\sigma_{k''} - k'\sigma_{k'})e^{i\theta_k} &= \mathcal{R} \sum_{p=k'+1}^{k''} s_p e^{i\theta_k} \\ &> \sum_{p=k'+1}^{k''} (-E + \rho | s_k |) = (k'' - k')(-E + \rho | s_k |) \end{aligned}$$

so that

$$(11.24) \qquad \rho \mid s_k \mid \leqq E + \mid k''\sigma_{k''} - k'\sigma_{k'} \mid /(k'' - k')$$

and our result follows easily by use of the crude inequality $| k''\sigma_{k''} - k'\sigma_{k'} | \leqq k'' | \sigma_{k''} | + k' | \sigma_{k'} |$.

This proof of Theorem 11.2 is so simple and straightforward as to be almost trivial. Indeed it may seem that the general class T''_E, at which we arrived after several successive generalizations of the original Tauberian class of series for which $nu_n \to 0$, is one designed especially to make possible a simple and transparent proof. It is a significant fact that direct proofs of C_1 Tauberian theorems for the class of series for which $nu_n \to 0$ are simple and straightforward; that such proofs for intermediate classes such as the class of series for which $n | u_n | < K$ or the classes T or T^* are more devious and complicated; and that finally such proofs for the larger classes T''_E are again simple and straightforward.

Several corollaries of Theorem 11.2 are easily obtained. In the first place, if $\sum u_n \in T''_E$ and σ_n is bounded, then s_n must be bounded. If it is true that not only $\sum u_n \in T''_E$ but also the series $(u_1 - A) + u_2 + u_3 + \cdots$ is in class T''_E for each A (and, by Theorem 10.1, in particular if $\sum u_n \in T^*$), then we can replace s_n and σ_n in (11.21) by $s_n - A$ and $\sigma_n - A$ respectively to obtain

$$(11.25) \qquad \limsup_{k\to\infty} | s_k - A | \leqq [E + F \limsup_{n\to\infty} | \sigma_n - A |]\rho^{-1}.$$

If $\sigma_n \to \sigma$ and (11.25) holds, then we can set $A = \sigma$ to obtain

$$(11.26) \qquad \limsup_{k\to\infty} | s_k - \sigma | \leqq E/\rho.$$

If it is also true that $\sum u_n \in T''$ and that E and ρ can be chosen such that E/ρ is arbitrarily near 0 (and in particular if $\sum u_n \in T'_\rho$ for some $\rho > 0$), then (11.26) implies that $\sum u_n$ converges to σ.

12. Conclusion. In conclusion, we use the preceding study of Tauberian conditions as a basis for indication that the general problem of obtaining Tauberian convergence and oscillation theorems may be divided naturally into

four categories. Order among these categories is not significant, since each category is necessary for a complete theory.

I. The first problem is that of discovery of significant classes S of series $\sum u_n$ of such a character that one may decide by inspection of the terms of $\sum u_n$ (and preferably with no knowledge whatever concerning properties of the sequence s_n of partial sums of $\sum u_n$) whether $\sum u_n \epsilon S$.

II. The second problem is that of devising criteria to assist in showing that $\sum u_n \epsilon S$.

III. The third problem is that of showing that S is a subclass of a general class G of series for which Tauberian oscillation theorems can be proved by straightforward methods. To provide for Tauberian convergence theorems, it is also desirable to prove that if $\sum u_n \epsilon S$, then for each complex A the series $(u_1 - A) + u_2 + u_3 + \cdots$ is in class G.

IV. The fourth problem is that of establishing Tauberian theorems for the classes G.

Problems of type IV are the ones which depend essentially upon the particular method of summability used. The attention of this paper has been centered almost exclusively on problems of the first three types.

Cornell University,
Ithaca, N. Y.

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ON DIVERGENCE PROPERTIES OF THE LAGRANGE INTERPOLATION PARABOLAS

By P. Erdös

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Throughout the present paper, $-1 < x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} < 1$ denote the roots of the n-th Tchebicheff polynomial $T_n(x)$, and unless otherwise stated it is understood that the fundamental points of the Lagrange interpolation are the $x_i^{(n)}$.[1] It is well known that[2] there exists a continuous function whose interpolation parabolas diverge everywhere in $(-1, +1)$. In the present paper we prove that for $x_0 = \cos \frac{p}{q}\pi_1$, $p \equiv q \equiv 1 \pmod 2$, $(p_1, q) = 1$ there exists a continuous function $f(x)$ such that $L_n f(x_0) \to \infty$.[3] Turán and I[4] proved that this does not hold for any other point. In this direction Marcinkievicz[5] proved that if the fundamental points are the roots of $U_n(x) = T'_{n+1}(x)$ then for every continuous function $f(x)$ and every point x_0 there exists a sequence of integers $n_1 < n_2 < \cdots$ such that $L_{n_i}(f(x_0)) \to f(x_0)$. We remark that in the case of the Fourier series it is well known that there always exists a subsequence of the partial sums converging to $f(x_0)$. This fact may be of interest because there is often an analogous behaviour of the Lagrange interpolation parabola and the Fourier series.

First we prove some lemmas.

Lemma 1.

$$x_i^{(m)} - x_j^{(n)} > \frac{1}{m^3}, \quad \textit{for } m \geqq n.$$

Proof. Write

$$x_i^{(m)} = \cos \vartheta_i^{(m)}, \qquad \vartheta_i = \frac{2i-1}{2m}\pi.$$

Then we have

$$|x_i^{(m)} - x_j^{(n)}| > |\vartheta_i^{(m)} - \vartheta_j^{(n)}| \sin \frac{\pi}{2n} > \frac{\pi}{4n}\frac{\pi}{2mn} > \frac{1}{m^3} \text{ q.e.d.}$$

[1] For the employed notations see P. Erdös and P. Turán, Annals of Math., Vol. 38 (1937), p. 142–155. If there is no danger of confusion we will omit the upper index n.

[2] G. Grünwald, Annals of Math., Vol. 37 (1936), p. 908–918.

[3] $L_n(f(x))$ denotes the Lagrange interpolation parabola of $f(x)$.

[4] This result was stated in the Annals of Math., Vol. 38 (1937), p. 155 but there was a misprint.

[5] Acta Litt ac Scient. Szeged, Tom. 8, p. 127–130.

LEMMA 2. *Put* $x_0 = \cos \frac{p}{q}\pi$, $p \equiv q \equiv 1 \pmod 2$; *then constants* c_1 *and* c_2 *exist such that*

$$\min_{i=1,2,\cdots n} |x_0 - x_i^{(n)}| > \frac{c_1}{n}, \qquad |T_n(x_0)| > c_2.$$

PROOF.

$$|T_n(x_0)| = \cos\left(\frac{np}{q}\pi\right) \geqq \cos\left(\frac{\pi}{2} - \frac{\pi}{2q}\right) > c_2.$$

Put $x_j^{(n)} < x_0 < x_{j+1}^{(n)}$; then

$$\min_{i=1,2,\cdots n} |x_0 - x_i^{(n)}| > \frac{\pi}{2nq}\min\left(\sin\frac{2j-1}{2n}\pi, \sin\frac{2j+1}{2n}\pi\right) > \frac{c_1}{n}.$$

LEMMA 3.

$$\sum{}' |l_k^{(n)}(x_0)| < (\log n)^{\frac{1}{2}}$$

where $\sum'$ *indicates that the summation is extended only over the* $x_k^{(n)}$ *satisfying* $|x_k^{(n)} - x_0| > \frac{1}{(\log n)^{\frac{1}{2}}}$.

PROOF.

$$|l_k^{(n)}(x_0)| = \left|\frac{T_n(x_0)}{T'_n(x_k)(x_0 - x_k)}\right| < \frac{(\log n)^{\frac{1}{2}}}{n}$$

since $|T_n(x_0)| \leqq 1$ and $T'_n(x_k) = \frac{n}{\sqrt{(1 - x_k^2)}} \geqq n$, which proves the Lemma. Without loss of generality we may assume that $x_0 > 0$. Let $x_j^{(n)} < x_0 < x_{j+1}^{(n)}$. Now we prove

LEMMA 4. *Suppose* $0 < x_k^{(n)} < x_j^{(n)}$ $\left(\text{i.e., } \frac{n}{2} < k < j\right)$; *then*

$$|l_k^{(n)}(x_0)| > \frac{c_3}{j-k}.$$

PROOF. We have

$$|l_k^{(n)}(x_0)| = \left|\frac{T_n(x_0)}{T'_n(x_k)(x_0 - x_k)}\right| \geqq \frac{c_2\sqrt{(1 - x_k^2)}}{n(x_0 - x_k)} > \frac{c_4}{n(x_{j+1} - x_k)},$$

by Lemma 2. Now $x_{j+1} - x_k < (j + 1 - k)\frac{\pi}{n} < \frac{c_5(j-k)}{n}$, which proves the Lemma.

LEMMA 5.

$$\sum_{(2k-1,n)=1} |l_k^{(n)}(x_0)| > c_6 \frac{\log n}{\log\log n}.$$

PROOF. By Lemma 4 we have

$$\sum_{(2k-1,n)=1} | l_k^{(n)}(x_0) | > \sum{}'' | l_k^{(n)}(x_0) | > c_3 \sum{}'' \frac{1}{j-k}$$

where the two dashes indicate that the summation is extended only over those k for which $(2k - 1, n) = 1$ and $\frac{n}{2} < k < j$. It is clear[6] that there are at least $c_7 n$ of the $x_k^{(n)}$ between 0 and $x_j^{(n)}$, thus

$$\sum{}'' \frac{1}{j-k} > \sum{}''' \frac{1}{j-k}$$

where the three dashes indicate that the summation is extended only over those k which satisfy $(2k - 1, n) = 1$ and $j - k < c_7 n$.

Denote by $\nu(n)$ the number of different odd prime factors of n. It is well known that $\nu(n) < c_8 \frac{\log n}{\log \log n}$. (This result is an immediate consequence of the prime number theorem, but can also be obtained in an elementary way.) The number of integers k satisfying $j - x < k < j$, $(2k - 1, n) = 1$ equals by the sieve of Eratosthenes

$$x - \sum_{p|n} \left[\frac{x}{p}\right]' + \sum_{pq|n} \left[\frac{x}{pq}\right]' - \cdots{}^{7} \geqq x \prod_{p|n} \left(1 - \frac{1}{p}\right) - 2^{\nu(n)}$$

$$> c_9 \frac{x}{\log \log n} - 2^{c_8 \log n / \log \log n} > c_{10} \frac{x}{\log \log n} \quad \text{for } x > \sqrt{n}, \quad (p \text{ odd})$$

since it is well known that $\prod_{p|n} \left(1 - \frac{1}{p}\right) > \frac{c_{11}}{\log \log n}$.[8] Thus

$$\sum{}''' \frac{1}{j-k} > \frac{c_{10}}{\log \log n} \sum_{c_7 n > r > \sqrt{n}} \frac{1}{r} > c_6 \frac{\log n}{\log \log n} \text{ q.e.d.}$$

THEOREM 1. *There exists a continuous function $f(x)$ such that $L_n(f(x_0)) \to \infty$.*

PROOF. Write

$$f(x) = \sum_{n=n_0}^{\infty} \frac{f_n(x)}{\sqrt{\log n}}.$$

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[6] i.e. $| x_{r+1}^{(n)} - x_r^{(n)} | \leqq \frac{\pi}{n}$, $r = 1, 2, \ldots$.

[7] $\left[\frac{x}{p}\right]'$ denotes the number of the k's in the interval $j - x < k < j$ for which $2k - 1$ is divisible by p. It is clear that $\left[\frac{x}{p}\right]'$ differs from $\frac{x}{p}$ by less than 1.

[8] E. Landau, *Über den Verlauf der zahlentheoretischen Function.* Archiv der Math. und Phys., Ser. 3, Vol. 5, (1903), p. 86–91.

$f_n(x)$ is defined as follows:

$$f_n(x_k^{(n)}) = \text{signum } l_k^{(n)}(x_0) \quad \text{for} \quad (2k-1, n) = 1,$$

$$f_n\left(x_k^{(n)} \pm \frac{1}{2^{2^n}}\right) = 0,$$

in the intervals $\left(x_k^{(n)}, x_k^{(n)} + \frac{1}{2^{2^n}}\right)$ and $\left(x_k^{(n)}, x_k^{(n)} - \frac{1}{2^{2^n}}\right)$, $f_n(x)$ is linear and elsewhere $f_n(x) = 0$.

First we show that $f(x)$ is continuous. It suffices to show that

$$\sum_{n=n_0}^{\infty} \frac{f_n(x)}{\sqrt{\log n}}$$

is uniformly convergent, i.e. that

$$\sum_{n>n(\epsilon)} \frac{f_n(x)}{\sqrt{\log n}} < \epsilon.$$

If for a certain y, $f_n(y)$ and $f_m(y)$, $m > n$ are both different from 0, we have for a certain k_1 and k_2

$$|x_{k_1}^{(n)} - y| < \frac{1}{2^{2^n}}, \qquad |x_{k_2}^{(m)} - y| < \frac{1}{2^{2^m}},$$

i.e.

$$|x_{k_1}^{(n)} - x_{k_2}^{(m)}| < \frac{2}{2^{2^n}}.$$

But by Lemma 1

$$|x_{k_1}^{(n)} - x_{k_2}^{(m)}| > \frac{1}{m^3}$$

hence $2m^3 > 2^{2^n}$, i.e. $m > n^2$ for $n > 3$. Thus

$$\sum_{n>n(\epsilon)} \frac{f_n(x)}{\sqrt{\log n}} < \sum_{r>r(\epsilon)} \frac{1}{\sqrt{\log 2^{2^r}}} < \epsilon.$$

Put

$$\varphi_1(x) = \sum_{r=n_0}^{n-1} \frac{f_r(x)}{\sqrt{\log r}}, \qquad \varphi_2(x) = \sum_{r>n} \frac{f_r(x)}{\sqrt{\log r}}.$$

Then

$$L_n(f(x_0)) = L_n(\varphi_1(x)) + L_n\left(\frac{f_n(x)}{\sqrt{\log n}}\right) + L_n(\varphi_2(x)).$$

First we show that $L_n(\varphi_2(x)) = 0$. It will evidently suffice to show that for every k, $\varphi_2(x_k^{(n)}) = 0$ or that for $r > n$, $f_r(x_k^{(n)}) = 0$. If for a certain $r > n, f_r(x_k^{(n)}) \neq 0$ we have for a certain l

$$| x_k^{(n)} - x_l^{(r)} | < \frac{1}{2^{2^r}},$$

which does not hold for by Lemma 1 for $2^{2^r} > r^3$.

Next we estimate $L_n(\varphi_1(x))$. If for a certain $x_k^{(n)}$, $f_r(x_k^{(n)}) \neq 0$ then for a certain l

$$| x_k^{(n)} - x_l^{(r)} | < \frac{1}{2^{2^r}}$$

which by Lemma 1 means that

$$2^{2^r} < n^3 \qquad \text{or} \qquad r < 2 \log \log n \text{ for } n > n_0 .$$

Thus if for a certain $x_k^{(n)}$, $\varphi_1(x_k^{(n)}) \neq 0$ then by Lemma 2

$$|x_k^{(n)} - x_0| > \min_{i=1,2,\cdots r} |x_i^{(r)} - x_0| - \frac{1}{2^{2^r}} > \frac{c_1}{r} - \frac{1}{2^{2^r}} > \frac{1}{(\log n)^{\frac{1}{2}}} \quad \text{for} \quad r > n_0.$$

Thus by Lemma 3

$$L_n(\varphi_1(x_0)) < c_{12} \sum_{|x_k - x_0| > (\log n)^{-\frac{1}{2}}} | l_k^{(n)}(x_0) | < c_{12}(\log n)^{\frac{1}{2}}$$

Now by Lemma 5

$$L_n(f_n(x_0)) = \sum_{(2k-1,n)=1} | l_k^{(n)}(x_0) | > c_6 \frac{\log n}{\log \log n}$$

since for $(2k - 1, n) \neq 1$ $f_n(x_0) = 0$. Thus finally

$$L_n(f(x_0)) > c_6 \frac{(\log n)^{\frac{1}{2}}}{\log \log n} - c_{12} (\log n)^{\frac{1}{2}} \to \infty .$$

Similarly we could prove that a continuous $f(x)$ exists such that $L_n(f(x_0))$ converges to any given value.

THEOREM 2. *If* $x_0 \neq \cos \frac{p}{q} \pi$, $p \equiv q \equiv 1 \pmod 2$ *then there exists for every continuous* $f(x)$ *a sequence of integers* $n_1 < n_2 < \cdots$ *such that* $L_{n_i}(f(x_0)) \to f(x_0)$.

PROOF. First we prove that there exists a sequence if integers $n_1 < n_2 < \cdots$ such that $| T_{n_k}(x_0) | < \frac{c_{13}}{n}$. We need the following

LEMMA 6. *If* $x_0 \neq \frac{p}{q}$, $p \equiv q \equiv 1 \pmod 2$, *then the inequality*

$$\left| x_0 - \frac{2r - 1}{2n_k} \right| < \frac{c_{14}}{n_k^2}$$

has an infinite number of solutions.

Proof. If x_0 is rational it is of the form $\frac{2r-1}{2n_k}$, thus the Lemma is trivial. Hence we may suppose that x_0 is irrational. It is well known that the equation $\left| x_0 - \frac{a}{b} \right| < \frac{1}{b^2}$ has an infinite number of solutions. If infinitely many of the b's are even the Lemma is proved, if not consider the least positive solution of

$$2a\,d - bf = 1.$$

Obviously $f \equiv 1 \pmod 2$ and $d < b$ thus

$$\left| x_0 - \frac{f}{2d} \right| \leqq \frac{1}{b^2} + \frac{1}{2bd} < \frac{c_{14}}{d^2}$$

which proves the Lemma.

If $\left| x_0 - \frac{2r-1}{2n_k} \right| < \frac{c_{14}}{n_k^2}$ we have

$$T_{n_k}(x_0) < \cos\left(\frac{\pi}{2} - \frac{c_{14}}{n_k}\right) < \frac{c_{13}}{n}.$$

Consider now a sequence of integers $n_1 < n_2 < \cdots$ with $\left| x_0 - \frac{2r-1}{n_k} \right| < \frac{c_{14}}{n_k^2}$. We are going to prove that $L_{n_k}(f(x_0)) \to f(x_0)$.

For $k \neq r$ we have

$$| l_k(x_0) | = \left| \frac{T_{n_k}(x_0)}{T'_{n_k}(x_k)(x_k - x_0)} \right| < \left| \frac{c_{13}}{n^2(x_k - x_0)} \right|.$$

Thus

$$\sum_{k \neq r} | l_k(x_0) | < \frac{c_{13}}{n^2} \sum_{k \neq r} \frac{1}{| x_k - x_0 |} = o(1),^9$$

hence from

$$\sum_{k=1}^{n} l_k(x) \equiv 1$$

---

[9] We have

$$\sum_{k \neq r} \frac{1}{x_k - x_0} = \sum\nolimits'_{|x_k - x_0| \leqq (\log n)^{-1}} \frac{1}{| x_k - x_0 |}$$

$$+ \sum\nolimits'_{|x_k - x_0| > (\log n)^{-1}} \frac{1}{| x_k - x_0 |} < n \log n + cn \log n = o(n^2).$$

(The dash indicates that $k = r$ is omitted.)

it follows that

$$l_r(x_0) = 1 - o(1).$$

Thus

$$L_{n_k}(f(x_0)) = f(x_r)l_r(x_0) + \sum_{k \neq r} f(x_k)l_k(x_0) = (f(x_0) + \epsilon)[1 - o(1)] + o(1) \to f(x_0),$$

which proves Theorem 2.

On the other hand we can prove that for every x in $(-1, +1)$ there exists a continuous $f(x)$ such that

$$\lim_{n\to\infty} \frac{\sum_{m \leq n} L_m(f(x_0))}{n} = \infty .$$

The proof is very similar to that of Theorem 1.

PRINCETON, N. J.

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ADDITIVE SET FUNCTIONS AND VECTOR LATTICES

BY S. BOCHNER AND R. S. PHILLIPS

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This paper arose out of an attempt to extend known results in the theory of countably additive set functions to the finitely additive case. The basic tool used in this investigation has been the vector lattice. We have found this approach helpful to the understanding of both additive set functions and vector lattices.

An important link between point functions and additive set functions is the theorem of Lebesgue that every set function which is absolutely continuous is the integral of a point function. Radon and Nikodym extended the theorem from ordinary Lebesgue measure to countably additive measure on general sets. If the measure is bounded this theorem states that in the Banach space of all absolutely continuous functions, step-functions are dense in the norm. It was shown recently by one of the authors (2) that the latter theorem remains valid for finitely additive Jordan measure in general; however in the course of the proof the more general case was reduced to the previous theorem of Lebesgue-Nikodym. In section II of the present note a new proof will be given. It will not presuppose any essential facts from the Lebesgue theory proper, being based on simple but important facts from the theory of vector lattices. These prerequisites are assembled in section I.

In section III we discuss some analogies between vector lattices and set functions; here again the approach rather than the result is new. The positive elements are shown to be additive set functions on the Boolean algebra of normal subspaces. As a consequence finitely additive set functions become completely additive on this algebra. It is significant to observe that this extension of the original algebra can also be expressed in terms of the stochastic distance. We further show that a vector lattice may be embedded in a second vector lattice having the same Boolean algebra of normal subspaces and possessing a unit element. Finally we consider briefly the question of a generalized base.

I. PROJECTIONS IN VECTOR LATTICES

A space L will satisfy the following five postulates:

I: L is a linear space with real scalars, and a relation $x > 0$ is defined on L.
II: If $x > 0$ and $y > 0$, then $x + y > 0$.
III: If $x > 0$ and α is a scalar, then $\alpha > 0$ implies $\alpha x > 0$ and conversely.
IV: Relative to the given order relation ($x > y$ means $x - y > 0$), L is a lattice.

We will write: $x \vee y$ for sup (x, y), $x \wedge y$ for inf (x, y); x^+ for $x \vee 0$, x^- for $-x \vee 0$, and $| x |$ for $x^+ + x^-$; the least upper bound (if it exists) of a set E will be denoted by $\bigvee_E x$ and the greatest lower bound by $\bigwedge_E x$.

V: If a set E is bounded above (below) then $\bigvee_E x (\bigwedge_E x)$ exists.

Proof of the following theorems can be found in a paper by Freudenthal (3): $x = x^+ - x^-$; $|x| = x \vee -x = x^+ \vee x^-$; $x \to a + x$ and $x \to \alpha x$ $(\alpha > 0)$ are lattice automorphisms; $x \to -x$ is a lattice anti-automorphism; $x \vee y + x \wedge y = x + y$; $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$; $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$; $\alpha > 0$ implies $\alpha \bigvee_E x = \bigvee_E \alpha x$, $\bigvee_{E,F} x \vee y = \bigvee_E x \vee \bigvee_F y$, $\bigvee_{E,F} x \wedge y = \bigvee_E x \wedge \bigvee_F y$, $\bigvee_{E,F}(x + y) = \bigvee_E x + \bigvee_F y$ (if the least upper bounds on the right side exist); and dual statements for the greatest lower bounds.

x will be said to be *disjoint* from y if $|x| \wedge |y| = 0$. Given a subset Y of L, Y' will be the set of all $x \in L$ disjoint from every $y \in Y$. Iterating the process, we define Y'' to be $(Y')'$ and similarly for Y''' etc.

As Y'' contains all elements of L disjoint from every element of Y', $Y'' \supset Y$. If $Y \supset Z$, then $Y' \subset Z'$. In particular $Y''' \subset Y'$. It follows that $Y' = Y''' = Y^v = \cdots$ and $Y'' = Y^{iv} = Y^{vi} = \cdots$. The sets Y and Y' have at most the null vector in common.

A subset E of L which contains along with x an element v such that $2|x| \leqq v$ will be called a *D-set.*

A subspace of L satisfying postulates I, II, III, IV, and V for sets bounded above (below) in L is called *normal*[1] if it contains with x all y such that $0 \leqq |y| \leqq |x|$. By a *direct decomposition* of L is meant a choice of disjoint complementary normal subspaces—that is, of normal subspaces S and T such that $S \wedge T = 0$ (null vector) and L is the direct sum of S and T.

If $x \geqq 0$ and if Y is a D-set, then we define

$$x_Y = \bigvee_{y \in Y} x \wedge y$$

to be the *projection* of x on Y. This is a fundamental notion for this paper. If Y is a normal subspace, then x_Y belongs to Y as it is the least upper bound of a set of elements in Y. If $x \geqq 0$ belongs to Y, then $x_Y = \bigvee_{y \in Y} x \wedge y = x$. Further if $x \geqq 0$, then $x - x_Y \geqq 0$.

The purpose of this section will be to show that Y' and Y'' define a direct decomposition of L and to study the case in which Y contains but a single element.

THEOREM 1. *If Y is any subset of L, then Y' is a normal subspace.*

Suppose y is a fixed element of Y and the x_i belong to Y'. Y' is linear because $|x| \wedge |y| = 0$ implies $|\alpha x| \wedge |y| = 0$ and $0 \leqq |x_1 + x_2| \wedge |y| \leqq (|2x_1| \vee |2x_2|) \wedge |y| = (|2x_1| \wedge |y|) \vee (|2x_2| \wedge |y|) = 0$. As $Y' \subset L$, II and III are valid in Y'. To establish IV and V let E be a subset of Y' bounded above by $z \in L$. Then by a repeated application of $(u \vee v + u \wedge v = u + v)$, we obtain

$$\begin{aligned} 0 \leqq |\bigvee_E x| \wedge |y| \leqq (\bigvee_E x^+) \wedge |y| &= \bigvee_E x^+ + |y| - \bigvee_E x^+ \vee |y| \\ &= \bigvee_E (x^+ \vee |y| + x^+ \wedge |y|) - \bigvee_E x^+ \vee |y| \\ &\leqq \bigvee_E (x^+ \vee |y|) + \bigvee_E (x^+ \wedge |y|) - \bigvee_E (x^+ \vee |y|) = 0. \end{aligned}$$

[1] Garrett Birkhoff (1) calls such a subspace a complete normal subspace; F. Riesz (6) denotes it by famille complète.

The dual statement for boundedness from below can be similarly shown. Finally if $|w| \leqq |x|$, then $0 \leqq |w| \wedge |y| \leqq |x| \wedge |y| = 0$.

LEMMA 1. *If Y is a D-set and if $x \geqq 0 \in L$, then $x - x_Y \in Y'$.*

Let $y \geqq 0$ belong to Y. Then

$$\begin{aligned} 0 \leqq (x - x_Y) \wedge y &\leqq (x - x_Y) \wedge x \wedge y \\ &\leqq (x - x_Y) \wedge x_Y \\ &= -x_Y + x \wedge 2x_Y = -x_Y + x \wedge 2\mathsf{V}_{y \in Y}(x \wedge y) \\ &= -x_Y + \mathsf{V}_{y \in Y}(x \wedge 2x \wedge y) \\ &= -x_Y + x_Y = 0. \end{aligned}$$

As an obvious consequence of the lemma we have

$$(x - x_Y)_Y = 0$$

and this justifies the name "projection."

THEOREM 2. *If Y is a normal subspace of L, then $Y = Y''$.*

We can restrict ourselves to positive elements since if $x \in Y$, then $x^+ \in Y$ and $x^- \in Y$. If $x \geqq 0 \in Y''$, then by the lemma the element $x - x_Y$ belongs to Y'. However, it also belongs to Y'' since $0 \leqq x - x_Y \leqq x$ and Y'' is normal. Our element must therefore be the null element, and hence $x = x_Y$. But x was an arbitrary element of Y'', and x_Y belongs to Y.

THEOREM 3. *If Y is a normal subspace of L, then Y and Y' define a direct decomposition of L, namely*

$$x = (x_Y^+ - x_Y^-) + (x_{Y'}^+ - x_{Y'}^-).$$

Any decomposition $x = x_1 + x_2$ where $x_1 \in Y$ and $x_2 \in Y'$ is certainly unique because Y and Y' are linear and have only the null vector in common. It is therefore sufficient to show that for $x \geqq 0$, $x = x_Y + x_{Y'}$. By lemma 1, $x - x_Y \in Y'$. Hence $x_{Y'} \geqq (x - x_Y)_{Y'} = x - x_Y$. Since $Y = Y''$, we likewise have $x_Y \geqq x - x_{Y'}$. Therefore $x = x_Y + x_{Y'}$.

Frédéric Riesz (6) has demonstrated theorems 1, 2, 3 for the special case where L is the space of linear functionals on a semi-vector lattice.

If Y consists of a single element y, then Y'' will be called a *principal normal subspace* and be designated by $P(y)$.

From the definition, $P(y) = (y)''$, it follows that $x \in P(y)$ if and only if $|z| \wedge |y| = 0$ implies $|z| \wedge |x| = 0$. Now $x \in P(y)$ and $|x| \wedge |y| = 0$ implies $|x| = |x| \wedge |x| = 0$. Hence if $x \in P(y)$ and if $x \neq 0$, then $|x| \wedge |y| > 0$. In Freudenthal's terminology (3), $|y|$ is a unit element of $P(y)$.

THEOREM 4. *A necessary and sufficient condition that x belong to $P(y)$ is that $x = \mathsf{V}_n x^+ \wedge n|y| - \mathsf{V}_n x^- \wedge n|y|$.*

If x is of this form, x clearly belongs to $P(y)$. To prove the converse we need only consider $x \geqq 0 \in P(y)$. As the set $\{n|y|\}$ is a D-set, it follows from

lemma 1 that $x - \bigvee_n x \wedge n \mid y \mid$ belongs to $P(y)'$. Being less than x, it also belongs to $P(y)$, and is therefore the null vector.

Theorem 4 shows that elements of $P(y)$ are approximable by elements bounded in $\mid y \mid$. The element y being fixed, boundedness in $\mid y \mid$ is the same as a Lipschitz condition in the case of point functions. If L is the space of additive set functions on a generalized Boolean algebra, $P(y)$ is the class of all functions absolutely continuous with respect to the set function y. Theorem 3 then gives a decomposition of the function into an absolutely continuous part and a singular part.

II. Absolutely Continuous Set Functions

We now consider a finitely additive Boolean algebra of subsets E of a given set G, and on it a (finitely additive) Jordan volume which will be denoted by $v(E)$ or $\mid E \mid$. We assume that $\mid G \mid = 1$.

We consider the Banach space V_1 of all set functions $F(E)$ of bounded variation relative to finite partitions $\delta = (E_\nu)$. We define $F > 0$ if $F(E) \geqq 0$ for all E and $F(E) > 0$ for some E. With this definition, V_1 has all properties I-V of a space L. The properties I–III are trivial. Let $[F_\alpha]$, α is an index, be a set of elements which are bounded from above by an element G. In order to show the existence of $\sup_\alpha F_\alpha$ we take for any set E any integer n, any set of indices $\alpha = \alpha_1, \cdots, \alpha_n$ (which need not be different), and any partition of E into disjoint sets $E_1, \cdots, E_n$, and we put

$$F(E) = \sup_{n,\alpha_\nu,E_\nu} (F_{\alpha_1}(E_1) + \cdots + F_{\alpha_n}(E_n)). \tag{1}$$

Since $F_{\alpha_\nu}(E_\nu) \leqq G(E_\nu)$ and $G(E)$ is additive, we obtain $F(E) \leqq G(E)$ and therefore $F(E)$ is finite. It is not hard to see that $F(E)$ is additive. Also putting $\alpha_1 = \cdots = \alpha_n = \alpha$ we obtain $F(E) \geqq F_\alpha(E)$. Lastly if $H(E) \geqq F_\alpha(E)$ for all α, then $H(E) = H(E_1) + \cdots + H(E_n) \geqq F_{\alpha_1}(E_1) + \cdots + F_{\alpha_n}(E_n)$ and therefore $H(E) \geqq F(E)$. The inf. can be obtained in a dual fashion. Thus properties IV and V are also satisfied.

An element F of V_1 is called absolutely continuous if for any $\epsilon > 0$ there exists an $\eta > 0$ such that $\mid E \mid < \eta$ implies $\mid F(E) \mid < \epsilon$. The totality of such elements F will be denoted by AC. Obviously $v(E) \in AC$. An element G of V_1 is called singular if to any $\epsilon > 0$ there exists a set E_ϵ for which $\mid E_\epsilon \mid < \epsilon$ such that for any E, $\mid G(E - E_\epsilon) \mid < \epsilon$. The totality of all such elements G will be denoted by S. It is easily seen that AC and S are each linear spaces with properties I—IV.

Now, if $F \in AC$ and $G \in S$ and $F \geqq 0$ and $G \geqq 0$, then

$$\inf (F, G) = \inf_{E'+E''=E}(F(E') + G(E'')) = 0.$$

Therefore

$$S \wedge AC = 0 \qquad \text{(null function)}$$

and

(2) $$S' \supset AC.$$

On the other hand if $G \geqq 0$, $G \epsilon V_1$, then

$$\inf (G, v) = \inf_{E'+E''=E}(G(E') + v(E'')) = 0$$

if and only if $G \epsilon S$. Therefore $S = (v)'$ and $(v)'' = S'$. Thus by (2) we obtain

(3) $$(v)'' \supset AC.$$

Our next step is to show that

(4) $$(v)'' \subset AC$$

and hence

$$(v)'' = AC = S'.$$

Now (4) asserts that any element $F \geqq 0$ for which

(5) $$F = \lim_{n\to\infty} \inf(F, nv)$$

is absolutely continuous (theorem 4). Each element $F_n = \inf (F, nv)$ belongs to AC since $0 \leqq F_n(E) \leqq n \mid E \mid$. Furthermore $F_n(E)$ converges toward $F(E)$ uniformly in E since $0 \leqq F(E) - F_n(E) \leqq F(G) - F_n(G)$, and therefore $F(E)$ also belongs to AC. By theorem 1, AC and S are normal subspaces.

It is now very easy to prove the following theorem:

THEOREM 5. *Step functions are dense in AC.*

Since for $F \geqq 0$,

$$|| F - F_n ||_1 = F(G) - F_n(G)$$

our argument shows that "bounded" functions are dense in AC in the norm of V_1, and so it suffices to prove that step functions are dense in the manifold of bounded functions. Introducing the space V_2 (see (2)), since the norm of a bounded element is larger in V_2 than it is in V_1 it is sufficient to prove the

LEMMA 2. *Step functions are dense in V_2.*

If $f(x)$ is a bounded integrable point function and if for some E we put

$$a = \frac{1}{|E|}\int_E f(x)\,dv$$

then obviously

$$\int_E |f(x) - a|^2\,dv = \int_E |f(x)|^2\,dv - |a|^2\cdot|E|.$$

We take a partition $\delta = (E_\nu)$, put $E = E_\nu$ and sum over ν. This will give

(6) $$||f - f_\delta||^2 = ||f||^2 - ||f_\delta||^2.$$

Now let F be an arbitrary element of V_2, δ a fixed partition, and δ' any partition $> \delta$. Since $F_{\delta'}$ (see (2)) is a step function, and $F_\delta = (F_\delta)_{\delta'}$ we obtain from (6)

$$\| (F - F_\delta)_{\delta'} \|^2 = \| F_{\delta'} \|^2 - \| F_\delta \|^2.$$

Taking the limit with respect to δ' we obtain by the definition of the norm

$$\| F - F_\delta \|^2 = \| F \|^2 - \| F_\delta \|^2.$$

Thus

$$\lim_\delta \| F - F_\delta \| = 0$$

and therefore F is the limit in the norm of step functions F_δ.

A Banach space X will be said to possess a *generalized base* if there exists a directed set of linear transformations $U_\delta(x)$ on X to a finite dimensional subspace of X such that (1) $\lim_\delta U_\delta(x) = x$ for all $x \in X$, and (2) $\| U_\delta \| \leqq M$ for all δ.

If X is the space AC and $\delta = (E_\nu)$ is a finite partition such that $v(E_\nu) \neq 0$ for any ν, then define

$$U_\delta(F) = \sum_\delta \frac{F(E_\nu)}{v(E_\nu)} \cdot v(E \cdot E_\nu).$$

It is clear from the proof of theorem 5 that $\lim_\delta U_\delta(F) = F$ in the norm topology. Further $\| U_\delta(F) \|_1 \leqq \| F \|_1$. Therefore AC possess a generalized base.

III. The Boolean Algebra of Normal Subspaces

The class **B** of all normal subspaces of a vector lattice L with properties I–V is a Boolean algebra. This known fact and further information concerning **B** will be derived in the present section from properties of projections.

In **B** the element L is the 1 element, the normal subspace consisting of only the null vector is the 0 element, and for a set $E = [A]$ of **B**, $\bigwedge_E A$ is the set intersection and $\bigvee_E A$ is the smallest element of **B** containing all elements A of E. Also if A is an arbitrary element of **B**, then the derived element A', as defined in section I, is the complement of A in **B** in the sense that $A \wedge A' = 0$, $A \vee A' = 1$, and $A'' = A$. It remains only to prove that **B** satisfies the distributive law. This will be done in theorem 8.

In the next three theorems, a fixed class of elements $[A_\alpha]$ of B will be considered, and as an abbreviation we will denote by P the normal subspace $\bigvee_\alpha A_\alpha$, and by R the set sum of all elements of the A_α. Obviously $P \supset R$, and therefore $P' \subset R'$ and $P'' = P \supset R''$. However since P is the smallest normal subspace containing R, $R'' \supset P$. Therefore $P = R''$ and $P' = R'$.

THEOREM 6. *If $[A_\alpha]$ is any class of elements of B, then*

$$(\bigvee_\alpha A_\alpha)' = \bigwedge_\alpha A_\alpha' \tag{7}$$

and

$$(\wedge_\alpha A_\alpha)' = \vee_\alpha A_\alpha'. \tag{8}$$

The relation $P' = R'$ implies that the following statements are equivalent: $x \epsilon (\vee_\alpha A_\alpha)'$; $| x | \wedge | a_\alpha | = 0$ for all elements of R; $x \epsilon A_\alpha'$ for all α; and $x \epsilon \wedge_\alpha A_\alpha'$. This proves (7). (8) follows from (7) by taking the complement on both sides and replacing A_α by A_α'.

THEOREM 7. *If $[A_\alpha]$ is any class of elements of* **B** *and if $x \geqq 0$, then*

$$x_{\vee A_\alpha} = \vee_\alpha x_{A_\alpha} \tag{9}$$

and

$$x_{\wedge A_\alpha} = \wedge_\alpha x_{A_\alpha}. \tag{10}$$

Now R is clearly a D-set. Therefore by lemma 1, $x - x_R \epsilon R' = P'$. Hence $x_{P'} = x - x_P \geqq x - x_R$. But $P \supset R$ implies $x_P \geqq x_R$, so that $x_P = x_R$. We have

$$x_R = \vee_{\alpha, A_\alpha} x \wedge a_\alpha = \vee_\alpha[\vee_{A_\alpha} x \wedge a_\alpha] = \vee_\alpha x_{A_\alpha}.$$

This proves (9). (10) follows from (9) in the following way:

$$\begin{aligned} x_{\wedge A_\alpha} = x_{(\vee A_\alpha')'} = x - x_{\vee A_\alpha'} &= x - \vee_\alpha x_{A_\alpha'} = x - \vee_\alpha(x - x_{A_\alpha}) \\ &= x - (x - \wedge_\alpha x_{A_\alpha}) = \wedge_\alpha x_{A_\alpha}. \end{aligned}$$

THEOREM 8. **B** *is a complete Boolean algebra. If $[A_\alpha]$ and* **B** *are elements of* **B**, *then*

$$(\vee_\alpha A_\alpha) \wedge B = \vee_\alpha(A_\alpha \wedge B)$$

and

$$(\wedge_\alpha A_\alpha) \vee B = \wedge_\alpha(A_\alpha \vee B).$$

In fact for $x \geqq 0$, we have by theorem 7

$$x_{\vee(A_\alpha \wedge B)} = \vee_\alpha x_{A_\alpha \wedge B} = \vee_\alpha(x_{A_\alpha} \wedge x_B) = \vee_\alpha x_{A_\alpha} \wedge x_B = x_{\vee A_\alpha} \wedge x_B = x_{(\vee A_\alpha) \wedge B}.$$

Before giving the next theorem we need a *definition*. The *sum* of any set E of non-negative elements of L will be defined as follows: For a given ordering x_α of E, suppose $\sum_\lambda x_\alpha$ to be defined for all $\lambda < \mu$. Define $\sum_\lambda x_\alpha = \vee_{\lambda<\mu}[\sum_\lambda x_\alpha] + x_\mu$. Continuing this process until the set is exhausted, obtains an element $\sum x_\alpha \epsilon L$. Suppose that x_α' is another ordering of E. We next show that $\sum x_\alpha' = \sum x_\alpha$. It is easily shown that for any finite set $\pi = (\alpha_1, \cdots, \alpha_n)$, $\sum_\pi x_\alpha' \leqq \sum x_\alpha$. Suppose now that for all $\lambda < \mu$, $\sum_\lambda x_\alpha' \leqq \sum x_\alpha - \sum_\pi x_\alpha'$ where π is any finite set of α's greater than or equal to μ. Hence $\vee_{\lambda<\mu}[\sum_\lambda x_\alpha'] \leqq \sum x_\alpha - \sum_\pi x_\alpha'$. As μ could also belong to π, this proves the induction. It follows that $\sum x_\alpha' \leqq \sum x_\alpha$. Similarly we could show that $\sum x_\alpha' \geqq \sum x_\alpha$. Our definition of sum therefore depends only on the set E. We will designate this sum by $\sum_E x$.

On the basis of the previous theorem 8 we now have the

THEOREM 9. *If $x \geqq 0$, then x_A defines a lattice homomorphism of* **B** *on L. In particular if $[A_\alpha]$ is an arbitrary class of disjoint elements of* **B**, *then*

$$x_{\vee A_\alpha} = \sum_\alpha x_{A_\alpha}.$$

THEOREM 10. *If L has a unit $y > 0$, that is $L = P(y)$, then the homomorphism which is defined by y_A (see theorem 9) is an isomorphism.*

Also, for each $A \in$ **B**, *$A = P(y_A)$ and thus $e = y_A$ can be written in the form $e = y_{P(e)}$. An element e can be written in the form y_A if and only if*

$$0 \leqq e \leqq y \quad \text{and} \quad e \wedge y - e = 0. \tag{11}$$

PROOF: By theorem 9, $A \rightarrow y_A$ is a homomorphism. Now if $A \rightarrow 0$, then $a \in A$, $a \geqq 0$ implies $y \wedge a = 0$. Since y is a unit this implies $a = 0$ and hence $A = 0$. Thus $A \rightarrow y_A$ is an isomorphism.

Since $y_A \in A$, we also have $P(y_A) \subset A$. Thus $P(y_A) = \varphi(A)$ is a function from B to B, and $\varphi(A)$ is always a (proper or improper) part of A. However, $\varphi(P(y_A)) = P(y_A)$, and if A_1, A_2 are disjoint then so are $\varphi(A_1)$ and $\varphi(A_2)$. Therefore $\varphi(A) = A$; that is $A = P(y_A)$. Finally (11) is fulfilled for $e = y_A$. If (11) holds then $e \in P(e) = A$ and $y - e \in A'$, so that $e = y_A$ by theorem 3.

As a corollary to theorem 10 we have

THEOREM 11. *If L has a unit $y > 0$, then the Boolean algebra B of normal subspaces A is isomorphic with the Boolean algebra E of elements e of L for which* (11) *holds.*

Consider a generalized finitely additive Boolean algebra T of elements τ, a finitely additive positive set function $y(\tau)$ on T, and the class L of all finitely additive set functions on T absolutely continuous with respect to $y(\tau)$. Our theory suggests a method of extending T modulo null sets to a sigma Boolean algebra on which $y(\tau)$ is completely additive. Now clearly y is a unit for $L = P(y)$. Each $\tau_0 \in T$ defines a function $y(\tau \cdot \tau_0)$ which in the notation of theorem 11 belongs to E. If $x > 0 \in L$, then it is easily shown that $x_{P(y(\tau \cdot \tau_0))} = x(\tau \cdot \tau_0)$. All sets of y-measure zero correspond to 0 in the Boolean algebra of normal subspaces. Finally it is necessary to add certain ideal elements of E which do not correspond to elements of T modulo null sets. By theorem 9 the set functions x_A are completely additive on B. In particular their values on the unit of T are completely additive. On sets which do correspond to τ_0, this value is precisely $x(\tau_0)$. It is clear that by treating x^+ and x^- in this way $x(\tau) \in L$ becomes a completely additive set function on the extended algebra E.

Kakutani has obtained a similar result, namely, the given Boolean algebra can be embedded in a sigma field of point sets such that the finitely additive function can be extended on this new field to be completely additive. The new field is in this case difficult to determine.

Now our extension modulo null sets of T to E is the smallest possible extension for which $y(\tau)$ becomes completely additive. For suppose T' were another such extension. In this case the elements of E' defined by (11) for the new lattice L' do correspond to elements of T'. As T' contains T modulo null sets, L' is lattice homomorphic with $L = P(y)$. Therefore $E' \supset E$.

We define the stochastic distance $\rho(\tau_1, \tau_2) = y(\tau_1 \cdot \tau_2^1) + y(\tau_1^1 \cdot \tau_2)$. If we complete T in the usual manner by taking Cauchy sequences, we obtain a sigma field which is an extension modulo null sets of T and on which y becomes completely additive. As the metric extension of a sigma field modulo null sets relative to a completely additive set function is precisely the original field modulo null sets, it follows that the metric extension is likewise the smallest possible extension modulo null sets of T. Therefore E is also the metric extension of T modulo null sets.

Now if L has no unit it is possible to obtain (by means of the choice axiom) a set of principal normal subspaces $A_\alpha = P(y_\alpha)$ such that $\bigvee_\alpha A_\alpha = L$, $A_\alpha \wedge A_\beta = 0$ if $\alpha \neq \beta$, and $y_\alpha > 0$. Putting $x_\alpha = x^+_{A_\alpha} - x^-_{A_\alpha}$ each element x of L can then be represented by the set of components $[x_\alpha]$, $x_\alpha \epsilon A_\alpha$. Not all possible combinations of components $[x_\alpha]$ will occur in the representation.

We now define a new vector lattice L^* to consist of *all* possible elements $x = [x_\alpha]$, $x_\alpha \epsilon A_\alpha$. Defining the lattice operations in the obvious way, the new lattice satisfies postulates I–V, possesses a unit element, namely $[y_\alpha]$, and the original lattice L can be embedded into it.

The Boolean algebra $\mathbf{B}^*$ of normal subspaces of L^* is lattice isomorphic with the Boolean algebra $\mathbf{B}$ of normal subspaces of L. $B^* \epsilon \mathbf{B}^*$ is the direct product of the normal subspaces $B_\alpha \subset A_\alpha$. We correspond $B^* \epsilon \mathbf{B}^*$ to $\bigvee_\alpha B_\alpha \epsilon \mathbf{B}$ and $A \epsilon \mathbf{B}$ to $[A \wedge A_\alpha] \epsilon \mathbf{B}^*$. By theorem 8, $(\bigvee_\alpha B_\alpha) \wedge A_\alpha = B_\beta$ and $\bigvee_\alpha (A \wedge A_\alpha) = A$. Hence this is a one to one correspondence. The correspondence clearly preserves order.

We wish finally to consider the question of a generalized base (see section II) in vector lattices which possess a norm relative to which they are complete. We suppose given a set of normal subspaces A_α each possessing a generalized base U^α_δ where the $\| U^\alpha_\delta \|$ are uniformly bounded. We suppose further that $\bigvee_\alpha A_\alpha = L$, $A_\alpha \wedge A_\beta = 0$ if $\alpha \neq \beta$, and that the norm of x $\eta(x) = \sum_\alpha \eta(x_{A_\alpha})$. These conditions are satisfied by vector lattices of type (L) (1) among which is the space V_1. It now follows (5) that L itself possesses a generalized base.

PRINCETON UNIVERSITY
THE INSTITUTE FOR ADVANCED STUDY

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FREE LATTICES[1]

BY PHILIP M. WHITMAN

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1. Introduction

A **lattice** (sometimes called a structure) is a partially ordered set of elements each two of which have a greatest lower and a least upper bound, denoted $A \cap B$ and $A \cup B$, read "A meet B" and "A join B." Postulates[2] for partial ordering are that $\leqq$, defined between some pairs of elements, satisfy

(1) $$A \leqq A \text{ for all } A;$$

(2) $$\text{if } A \leqq B \text{ and } B \leqq A, \text{ then } A = B;$$

(3) $$\text{if } A \leqq B \text{ and } B \leqq C, \text{ then } A \leqq C.$$

A lattice is said to be **generated** by a set of elements X_i (the "generators") if it consists of the X_i and their finite combinations by $\cap$ and $\cup$—e.g., $X_1 \cup X_2$, $\{[X_1 \cup X_2] \cap X_3\} \cup X_1$—sometimes known as "lattice polynomials."

However, these polynomials do not all constitute distinct elements; for instance, $X_i \cap X_i = X_i$ in any lattice by definition of $\cap$; furthermore, in a specific lattice we may have additional rules of equality; e.g. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

The **free lattice** generated by the X_i is[3] a lattice generated by them in which there are no laws of equality except those derivable from the postulates for a lattice. This is the most general lattice generated by the X_i, in the sense that every other can be obtained from it by a homomorphism—determined by the additional rules of equality.

We are concerned here with the internal structure of a free lattice. Given two polynomials, how can we find which of $\leqq$, $=$, $\geqq$ (if any) hold between them in the free lattice (equivalently, in all lattices)? This question is answered in §2. In §3 we show that given a polynomial, there is a shortest polynomial

[1] Presented to the American Mathematical Society Sept. 5, 1939.

[2] Equivalently, we may postulate for a lattice the identities $L1$: $X \cup X = X \cap X = X$; $L2$: $X \cup Y = Y \cup X$, $X \cap Y = Y \cap X$; $L3$: $X \cup (Y \cup Z) = (X \cup Y) \cup Z$, $X \cap (Y \cap Z) = (X \cap Y) \cap Z$; $L4$: $X \cup (X \cap Y) = X \cap (X \cup Y) = X$, note the theorem that $X = X \cap Y$ if and only if $Y = X \cup Y$, and define $X \leqq Y$ as equivalent to $X = X \cap Y$ or $Y = X \cup Y$. Cf. Ore, *On the Foundation of Abstract Algebra I*, Annals of Math., 36 (1935), p. 409.

[3] Its existence is guaranteed by a theorem of universal algebra; cf. G. Birkhoff, *On the Structure of Abstract Algebras*, Proc. Camb. Phil. Soc., 31 (1935), pp. 440–1. Compare free groups.

equal to it in the free lattice (which can be found by a definite procedure), and in §4 it is shown that certain elements cover others.

The author is indebted to Prof. Garrett Birkhoff for many helpful suggestions.

2. Conditions for $A \leqq B$ in a free lattice

Given two elements A and B, we should like to know: is $A \leqq B$ in the free lattice? Sometimes this is obviously true; e.g. $X_1 \leqq X_1 \cup X_2$. Sometimes we can show it false, for by the abovementioned homomorphism, $A \leqq B$ in the free lattice implies $A \leqq B$ in every lattice with the same generators. Hence if we can exhibit a specific lattice where $A \nleqq B$, then $A \nleqq B$ in the free lattice. But this is a trial and error method; we should like some definite procedure for settling the question.

THEOREM 1. *In the free lattice generated by a set of elements* X_i,

(4) $X_i \leqq X_j$ if and only if $i = j$;

(5) *recursively,* $A \leqq B$ *if and only if one or more of the following hold:*

(a) $A \equiv A_1 \cup A_2$ *where* $A_1 \leqq B$ *and* $A_2 \leqq B$,

(b) $A \equiv A_1 \cap A_2$ *where* $A_1 \leqq B$ *or* $A_2 \leqq B$,

(c) $B \equiv B_1 \cup B_2$ *where* $A \leqq B_1$ *or* $A \leqq B_2$,

(d) $B \equiv B_1 \cap B_2$ *where* $A \leqq B_1$ *and* $A \leqq B_2$.

NOTE. In (5*a*) it is permissible that A_1 be itself a join; etc.

We see that this is the sort of condition desired, for, given A and B, we need look only at B and part of A, and A and part of B. Since the elements are the *finite* combinations of the generators, this process will eventually end with (4).

PROOF. These conditions are obviously sufficient; to prove them necessary we proceed by a series of definitions and lemmas.

DEFINITION OF $\subset$. **(6)** $X_i \subset X_j$ if and only if $i = j$; **(7)** $A \subset B$ if and only if one or more of (5*a*–*d*) are true with $\leqq$ replaced by $\subset$.

(8) DEFINITION. $A \supset B$ if and only if $B \subset A$.

NOTE. The set of definitions (6)–(8) is self-dual; i.e., if $\cap$ and $\cup$, $\subset$ and $\supset$ are interchanged the set remains the same. In particular, (7*a*) and (7*d*) are dual, (7*b*) and (7*c*) are dual, and (6) is self-dual. This property will enable us to omit many cases in proofs, where we need only make these same changes throughout.

(9) DEFINITION. $A \cong B$ if and only if $A \subset B$ and $A \supset B$.

(10) DEFINITION. The **length** of A, denoted $L(A)$, is the total number of X's appearing in A, counting repetitions; e.g., $L(X_1) = 1$, $L(X_1 \cup X_1) = 2$, $L(\{[X_1 \cup X_2] \cap X_3\} \cup X_1) = 4$.

With $\leqq$ as inclusion relation and $=$ as equality the combinations of the X_i form the free lattice; we now show that they also form a lattice with $\subset$ and $\cong$ in these roles; cf. (16).

(11) LEMMA. **(a)** $A \subset A$, **(b)** $A \subset A \cup B$, **(c)** $A \supset A \cap B$ *for all* A, B. *Thus* $A \cup B$ *is an upper bound to* A *and* B *under* $\subset$.

PROOF by induction on $L(A)$. (11*a*) is true for $L(A) = 1$, by (6). If (11*a*) is true for $L(A) \leqq m$, then so are (11*b*, *c*) by (7*c*, *b*). Hence (11*a*) holds for

$L(A) = m + 1$, for say $A \equiv A_1 \cup A_2$ (dually if $A \equiv A_1 \cap A_2$); then $A_1 \subset A$, $A_2 \subset A$ by induction; $\therefore A \equiv A_1 \cup A_2 \subset A$ by (7*a*).

(12) LEMMA. (a) *$A_1 \cap A_2 \subset X_j$ if and only if $A_1 \subset X_j$ or $A_2 \subset X_j$.*
(b) *$X_j \subset B_1 \cup B_2$ if and only if $X_j \subset B_1$ or $X_j \subset B_2$.*
(c) *$A_1 \cup A_2 \subset B$ if and only if $A_1 \subset B$ and $A_2 \subset B$.*
(d) *$A \subset B_1 \cap B_2$ if and only if $A \subset B_1$ and $A \subset B_2$.*
(e) *$A_1 \cap A_2 \subset B_1 \cup B_2$ if and only if $A_1 \cap A_2 \subset B_1$ or $A_1 \cap A_2 \subset B_2$ or $A_1 \subset B_1 \cup B_2$ or $A_2 \subset B_1 \cup B_2$.*

PROOF. "If" is obviously true by (7). "Only if" is proved by induction on $m = L(A) + L(B)$. It is true for $m \leqq 2$ vacuously. Assume (12) true for $m \leqq k - 1$; then for $m = k$,

(12*a*) $A_1 \cap A_2 \subset X_j$. Then $A_1 \subset X_j$ or $A_2 \subset X_j$, as desired, by (7*b*), the only part of (7) which applies.

(12*c*) $A_1 \cup A_2 \subset B$. *Case* 1. $A_1 \cup A_2 \subset X_j$. Then $A_1 \subset X_j$, $A_2 \subset X_j$ by (7*a*). *Case* 2. $A_1 \cup A_2 \subset B_1 \cup B_2$. Then by (7*a*, *c*), (*i*) $A_1 \subset B$ and $A_2 \subset B$ or (*ii*) $A_1 \cup A_2 \subset B_1$ or (*iii*) $A_1 \cup A_2 \subset B_2$. If (*ii*) $A_1 \cup A_2 \subset B_1$, then $A_1 \subset B_1$ and $A_2 \subset B_1$ by induction (12*c*); $\therefore A_1 \subset B_1 \cup B_2$, $A_2 \subset B_1 \cup B_2$ by (7*c*), as desired. Likewise if (*iii*) holds the lemma does, and if (*i*), then the proof is immediate. *Case* 3. $A_1 \cup A_2 \subset B_1 \cap B_2$. Then (*i*) $A_1 \subset B_1 \cap B_2$ and $A_2 \subset B_1 \cap B_2$ or (*ii*) $A_1 \cup A_2 \subset B_1$ and $A_1 \cup A_2 \subset B_2$, by (7*a*, *d*). If (*i*), Q.E.D. If (*ii*), then $A_1 \subset B_1$, $A_2 \subset B_1$, $A_1 \subset B_2$, $A_2 \subset B_2$ by induction (12*c*); $\therefore A_1 \subset B_1 \cap B_2$, $A_2 \subset B_1 \cap B_2$ by (7*d*).

(12*b*, *d*) dually.

(12*e*) $A_1 \cap A_2 \subset B_1 \cup B_2$. Then $A_1 \subset B_1 \cup B_2$ or $A_2 \subset B_1 \cup B_2$ or $A_1 \cap A_2 \subset B_1$ or $A_1 \cap A_2 \subset B_2$ by (7*b*, *c*). Q.E.D.

(13) LEMMA. *If $A \subset B$ and $B \subset C$, then $A \subset C$.*

PROOF by induction on $m = L(A) + L(B) + L(C)$. True for $m = 3$ by (6). Induction:

Case 1. *Non-meet $\subset B \subset C$.*

Case 1*a*. $X_i \subset X_j \subset C$. $\therefore i = j$ by (6). $\therefore X_i \equiv X_j$. $\therefore X_i \subset C$.

Case 1*b*. $X \subset B_1 \cup B_2 \subset C$. $\therefore X \subset$ some B_i by (12*b*). $B_i \subset C$ by (12*c*). $\therefore X \subset C$ by induction.

Case 1*c*. $X_i \subset B_1 \cap B_2 \subset X_j$. Part of dual of case 1*b*.

Case 1*d*. $X \subset B_1 \cap B_2 \subset C_1 \cup C_2$.

$\therefore B_1 \cap B_2 \subset$ some C_i or some $B_i \subset C_1 \cup C_2$ by (12*e*).

$\therefore X \subset C_i$ by induction. $X \subset B_i$ by (12*d*).

$\therefore X \subset C_1 \cup C_2$ by (12*b*) or (7*c*). $\therefore X \subset C_1 \cup C_2$ by induction.

Case 1*e*. $A_1 \cup A_2 \subset B \subset C$. $\therefore A_i \subset B$ (all i) by (12*c*). $\therefore A_i \subset C$ (all i) by induction. $\therefore A_1 \cup A_2 \subset C$ by (12*c*).

Case 1*f*. $X \subset B_1 \cap B_2 \subset C_1 \cap C_2$. Part of dual of case 1*e*.

Case 2. *Meet $\subset B \subset C$.*

Case 2*a*. $A_1 \cap A_2 \subset B \subset$ non-join. Part of dual of case 1.

Case 2*b*. $A_1 \cap A_2 \subset X \subset C_1 \cup C_2$. $\therefore$ some $A_i \subset X$ by (12*a*). $\therefore A_i \subset C_1 \cup C_2$ by induction. $\therefore A_1 \cap A_2 \subset C_1 \cup C_2$ by (12*e*).

Case 2*c*. $A_1 \cap A_2 \subset B_1 \cup B_2 \subset C_1 \cup C_2$.

$\therefore A_1 \cap A_2 \subset$ some B_i or some $A_i \subset B_1 \cup B_2$ by (12*e*).

$B_i \subset C_1 \cup C_2$ by (12*c*). $\therefore A_i \subset C_1 \cup C_2$ by induction.

$\therefore A_1 \cap A_2 \subset C_1 \cup C_2$ by induction. $\therefore A_1 \cap A_2 \subset C_1 \cup C_2$ by (12*e*).

Case 2*d*. $A_1 \cap A_2 \subset B_1 \cap B_2 \subset C_1 \cup C_2$. Dual of case 2*c*.

(14) Lemma. *$A \cup B$ is the least upper bound to A and B under $\subset$; dually for $A \cap B$.*

For it is *an* upper bound by (11), and if $A \subset C$ and $B \subset C$, then $A \cup B \subset C$ by (7*a*).

(15) Lemma. *$\cong$ is an equality relation.*[4]

(16) Lemma. *The finite combinations of the X_i by $\cap$ and $\cup$ form a lattice, generated by the X_i, with $\subset$ as inclusion relation and $\cong$ as equality and $A \cup B$, $A \cap B$ as least upper, greatest lower bounds to A and B.*

Proof. (1), (2), (3) are satisfied [(11), (9), (13)], and $A \cup B$ is least upper bound by (14).

Proof of theorem 1. Any other lattice generated by the X_i is a homomorphic image of the free lattice, hence $\leqq$ in a free lattice is sufficient for $\subset$. But by definition of $\cup$ as least upper bound and induction, $\subset$ is sufficient for $\leqq$ and hence $\cong$ for $=$. Therefore $\subset$ and $\leqq$ are equivalent in the free lattice. Theorem 1 then follows from (6) and (7).

Now denote $A_1 \cup A_2 \cup \cdots \cup A_n$ by $\sum_1^n A_i$, $A_1 \cap \cdots \cap A_n$ by $\prod_1^n A_i$, or simply $\sum A_i$, $\prod A_i$ if there is no confusion.

(17) Corollary. **(a)** *$\prod A_i \subset X_j$ if and only if some $A_i \subset X_j$.*

(b) *$X_i \subset \sum B_j$ if and only if $X_i \subset$ some B_j.*

(c) *$\sum A_i \subset B$ if and only if every $A_i \subset B$.*

(d) *$A \subset \prod B_j$ if and only if $A \subset$ every B_j.*

(e) *$\prod A_i \subset \sum B_j$ if and only if $\prod A_i \subset$ some B_j or some $A_i \subset \sum B_j$.*

(f) *$A_i \subset \sum A_j$, $B_i \supset \prod B_j$ all i.*

(g) *$\sum A_i$ is the least upper bound to $A_1, \cdots, A_n$ under $\subset$.*

Proof. By repetition of (12), (11).

To answer "Is $A = B$?," apply (2) and theorem 1. Conditions (12) or (17) are more convenient in practise than (5). (17*b*) is like the condition that a prime number divide a product.

3. Canonical forms

Having thus found one collection of elements equal to each other, and likewise other collections, we should like to choose as canonical forms one element from each collection.

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[4] Cf. Schröder, *Algebra der Logik*, vol. 1, p. 184, or MacNeille, *Partially Ordered Sets*, Trans. Am. Math. Soc., 42 (1937), pp. 416–60, or it may be readily verified directly.

THEOREM 2. *Of all the elements equal in a free lattice, there is one of shortest length, unique except for commutativity and associativity.*

PROOF. We show by induction on $k = L(A) + L(B)$ that if $A \cong B, A \not\equiv B$, then $C \exists: A \cong B \cong C$, and $L(C) < L(A)$ or else $L(C) < L(B)$. Then if the theorem were false, say A and B were alleged to be both of the shortest length, we could find a still shorter element unless $A \equiv B$! It is true vacuously for $k = 2$. Induction:

Case 1. $\sum A_i \cong \sum B_i$. Here, but not in the previous section, we assume A_i and B_i not themselves joins. $\sum A_i \subset \sum B_i$ by (9). $\therefore A_i \equiv \prod_j a_j^i \subset \sum B_n$ (all i) by (17*c*). $\therefore$ by (17*e*), for any i, (*i*) some $a_p^i \subset \sum B_j$ or (*ii*) $A_i \subset$ some B_r. Note that (*ii*) holds if $A_i \equiv X_s$, by (17*b*). Similarly, for any i, (*iii*) some $b_p^i \subset \sum A_j$ or (*iv*) $B_i \subset$ some A_r.

If (*i*) holds for some i, then $a_p^i \subset \sum B_j \subset \sum A_j$ by (9). $A_j \subset \sum A_n$, $j \neq i$. $\therefore a_p^i \cup \sum_{j\neq i} A_j \subset \sum A_j$. Also $A_i \subset a_p^i$, so $\sum A_j \subset a_p^i \cup \sum_{j\neq i} A_j$; $\therefore a_p^i \cup \sum_{j\neq i} A_j \cong \sum A_j$ and the theorem holds in this case; likewise if (*iii*) holds for some i.

Otherwise, (*ii*) and (*iv*) hold for all i. $\therefore$ for all i, $A_i \subset$ some $B_{f(i)}$, $B_i \subset$ some $A_{g(i)}$ [f, g need not be single-valued]. If $i \exists: g[f(i)] \neq i$, then $A_i \subset B_{f(i)} \subset A_j$ $(j \neq i)$, $\sum_{n\neq i} A_n \cong \sum A_n$, and the theorem holds; similarly if $f[g(i)] \neq i$. If not, then $f[g(i)] = i$, $g[f(i)] = i$ for all i. $\therefore A_i \cong B_{f(i)}$, $B_i \cong A_{g(i)}$ all i. But $\sum A_i \not\equiv \sum B_i$ by hypothesis, $\therefore p \exists: A_p \not\equiv B_{f(p)}$ or $B_p \not\equiv A_{g(p)}$, say the former. $\therefore$ by induction, $D \exists: A_p \cong B_{f(p)} \cong D$, $L(D) < L(A_p)$ or $L(D) < L(B_{f(p)})$, say the former. Then $D \cup \sum_{n\neq p} A_n \cong \sum A_n$ and the theorem holds.

Case 2. $\prod A_i \cong \prod B_i$. Dual.

Case 3. $\sum A_i \cong \prod B_j$. $\therefore \sum A_i \subset \prod B_j$; hence every $A_i \subset \prod B_j$ and $\sum A_i \subset$ every B_j. Also $\prod B_j \subset \sum A_i$,

$$\therefore \prod B_j \subset \text{some } A_p \quad \text{or some } B_p \subset \sum A_i.$$

$$A_p \subset \prod B_j \text{ by above.} \qquad \sum A_i \subset B_p \text{ by above.}$$

$$A_p \cong \prod B_j \cong \sum A_i. \qquad \therefore B_p \cong \sum A_i \cong \prod B_j. \qquad \text{Q.E.D.}$$

We take this unique shortest form as the canonical form.

(18) COROLLARY 1. $A \equiv \sum A_i \equiv \sum_i (\prod_j a_j^i) \cup \sum_{i\in E} X_i$ (E any subset of $1, \cdots, n$) is canonical if and only if (*a*) no $a_p^i \subset \sum A_n$ and (*b*) no $A_i \subset \sum_{n\neq i} A_n$ and (*c*) every A_i is canonical. Dually for $\prod A_i$.

Corollary 1 follows from the proof of theorem 2. From case 3 we might also require: no $A_p \cong \sum A_n$, but this is included in (*b*).

If $\sum A_i$ is not canonical, we can find the canonical element equal to it, for if (*a*) is false, then $a_p^i \cup \sum_{j\neq i} A_j \cong \sum A_j$, if (*b*) is false, then $\sum_{n\neq i} A_n \cong \sum A_n$, if (*c*) is false then replace it by its canonical form; in any case we get a shorter, equal element, to which we can again apply the process.

We can now build up a diagram of the free lattice step by step, starting with

the shortest elements. We could not hope to get it all at once since there are infinitely many distinct elements if there are more than two generators.[5]

(19) Corollary 2. If $\sum A_i \cong \sum B_i$, and $\sum A_i$ is canonical, then $(a) B_i \subset \sum A_i$, all i, and (b) given $i, j \ni: A_i \subset B_j$.

Proof. (a) follows from (12c), and if $A_i \equiv X$, then (b) follows from (12c, b). Otherwise $A_i \equiv \prod_j a_j^i \subset \sum B_i$ by (12c). $\therefore$ by (12e), either $A_i \subset$ some B_j as desired, or some $a_j^i \subset \sum B_i \cong \sum A_i$ and then by (18a) $\sum A_i$ is not canonical contrary to hypothesis.

4. Covering theorems

Definition. A **covers** B if $A > B$ and no $C \ni: A > C > B$.

We present some scattered results on the existence of such pairs of elements. Denote the free lattice of n generators by F_n. In this section (but not previously) we assume n finite.

Theorem 3. *In the free lattice of n generators, if* $A > (\sum_{i \neq p} X_i) \cap (\sum_{i \neq q} X_i)$, *then* $A = \sum_1^n X_i$ or $\sum_{i \neq p} X_i$ *or* $\sum_{i \neq q} X_i$.

We first prove a

Lemma. *In* F_n, $A \equiv \sum A_i > \sum_{i \neq p} X_i$ *if and only if given* $i, j \ni: A_j \geqq X_i$.

Proof of lemma by induction on $L(A)$. Obvious for $L(A) \leqq n$. Suppose $L(A) = k > n$, obviously redundant terms having been omitted; e.g., $X_1 \cup X_1$ contracted to X_1. Also, by (17b), we may suppose no A_i is itself a join. Then some A_i has at least two factors, say $A_1 \equiv \prod a_j$. Then

$$\sum_{i \neq p} X_i < \sum A_i \leqq a_s \cup \sum_{i \neq 1} A_i \qquad \text{(all } s\text{)}.$$

By induction, given $i, j \ni$: j^{th} term $\geqq X_i$. If for some s this term is an A_i, then the lemma holds; otherwise $a_s \geqq X_i$ for all s, and $A_1 \geqq X_i$.

Corollary 1. *In* F_n, $\sum A_i = \sum_1^n X_i$ *if and only if given* $i, j \ni: A_j \geqq X_i$. $\sum A_i = \sum_{i \neq p} X_i$ *if and only if given* i $(i \neq p)$, $j \ni: A_j \geqq X_i$, *but no* $A_j \geqq X_p$.

Corollary 2. *In* F_n, $\sum_1^n X_i$ *covers* $\sum_{i \neq p} X_i$.

Corollary 3. *In* F_n, *every* $A \neq \sum_1^n X_i$ *is* $\leqq \sum_{i \neq p} X_i$ *for some p.* Proved by induction.

The rest of theorem 3 is now proved in much the same way.

Corollary 4. *If* F_n, $\sum_{i \neq p} X_i$ *covers* $(\sum_{i \neq p} X_i) \cap (\sum_{i \neq q} X_i)$, *any q.*

Corollary 5. *In* F_n, *if* $A > X_p$, *then* $A \geqq X_p \cup (\prod_{i \neq p} X_i)$.

Theorem 4. $\sum_1^n X_i$ *covers* $\sum_{i \neq p} X_i$ *which in turn covers* $(\sum_{i \neq p} X_i) \cap (\sum_{i \neq q} X_i)$, *and* $X_p \cup (\prod_{i \neq p} X_i)$ *covers* X_p, *in any lattice generated by the* X_i *in which these elements are distinct.*

For any other lattice is a homomorphic image of the free one.

The duals of these results are of course likewise true.

Harvard University

[5] Cf. G. Birkhoff, *op. cit.*, p. 451. The distinctness of his elements in the free lattice can also be shown by (12). Note that, taking every sixth element, we get an infinite "chain" of distinct elements.

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CORRECTIONS TO OUR PAPER ON THE EXISTENCE OF MINIMAL SURFACES OF GENERAL CRITICAL TYPES

By Marston Morse and C. Tompkins

The following corrections should be made in our paper which appeared in volume 40 (1939) of these Annals.

(1). In equation (5.14) interchange $\varphi(\beta)$ and $\psi(\alpha)$.

(2). The argument on page 454 following equation (5.16) should be replaced by the following. The case $\alpha = \beta$ in (5.16) may be discarded. When $\alpha \neq \beta$ (5.16) occurs only when $\psi(\alpha)$ is constant on some interval. In this case the harmonic surface defined by $\psi(\alpha)$ is not a minimal surface (see Radó, loc. cit., p. 75). The transformations used in §6 suffice in this case to prove that $A(\varphi)$ is upper reducible at ψ.

(3). Replace the first two paragraphs on page 460 by the following. A directly conformal 1-1 transformation of the disc $r \leqq 1$ into itself induces a transformation $T(\theta)$ of the circle $r = 1$ into itself. A curve $[p]$ of Z will be said to be *equivalent* to $\varphi(\alpha)$ if for a suitably chosen transformation $T(\theta)$ of the above type, the transformation $\varphi(T(\theta))$ "defines" $[p]$.

(4). In the fourth line following Lemma 6.1 replace "the point $f(p)$ of Ω" by a point $f(p)$ of Ω equivalent to $[p]$ and varying continuously with $[p]$.

The Institute for Advanced Study,
Princeton University.